TRANSIENTS IN ELECTRIC CIRCUITS

USING THE

HEAVISIDE OPERATIONAL CALCULUS

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BY

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PREFACE

THE study of electrical transients is a branch of the study of vibrations, and has always been of great mathematical interest. With recent developments in engineering it has now become of great practical importance. Certain branches of engineering, as for example communication engineering, are essentially the study of controlled transients.

This book deals especially with electrical engineering problems. The formal mathematical treatment has been discarded, and the Heaviside operational methods, now widely used in technical literature, adopted. I do not profess to have given a rigorous treatment of operational methods, but reference to the writings of Bromwich, Jeffreys, Carson, and others will readily supply this deficiency.

Actually, the work may be regarded as consisting of two main sections. The first section, Chapters I–VI, gives the theory of lumped circuits, wherein the expansion yields negative powers of the operator p. The second section deals with smooth circuits, or repeated lumped circuits, wherein expansion gives fractional positive powers of p. Problems of the latter section are analogous to those of weighted strings. On writing down a differential equation for such a string the tacit assumption is made that dy/dx is continuous, but this need not be the case. To avoid this difficulty the modern method is to view the system as the limiting conditions of a discrete system. This method has been adopted.

In Chapter X I have discussed Fourier series and Fourier integrals in order to give further insight into operational methods and to show that the Fourier integral yields a long list of operators and their equivalencies.

Some of the methods of dealing with variable circuits have been discussed in Chapters XI and XII. This is a very wide subject, and considerably more mathematical research is needed to put it on a firmer (i.e. more rigorous) basis.

Throughout the work the technical press has been freely drawn upon for practical illustrative matter. In several investigations, simplifying assumptions have been made in order to bring the work within the compass of this book. Indebtedness

to the writings of others is evident from the many references in the text.

I am also indebted to my colleague, Assistant Professor W. O. Richmond, M.A.Sc., for reading the manuscript and for several helpful criticisms, and also to the publishers for their painstaking work.

W. B. C.

Vancouver

CONTENTS

TRANSIENTS IN ELECTRIC CIRCUITS

CHAPTER I

GENERAL THEORY

1. **Definitions.** Transient conditions refer to what happens between steady-state conditions when the initial stability is disturbed. They are usually of short duration and of decreasing amplitude with respect to time, space, or both. In engineering practice, studies of these conditions are of increasing importance; for example, communication engineering is in reality the study of controlled transients.

It is usual to segregate transients as *free* or *forced*. In the free condition, the transient is occasioned by switching or some such change of circuit conditions. The duration of the disturbance will depend on circuit constants. In the forced condition the transient is set up by induction, radiation, etc., and the period of the disturbance will depend on the source creating the disturbance. In the solution of differential equations the *particular integral* yields the forced vibration, while the *complementary integral* gives the free vibration.

2. **Representation.** The mathematical representation of a transient term will now be explained. Every engineer is familiar with the fact that a harmonically varying quantity may be represented by $A \sin \omega t$, and that such a quantity may be represented as a vector of length A rotating in a plane about the origin with angular velocity $2\pi f$, where f is the frequency.

For transient conditions the quantity varies with respect to time or space exponentially. Then the length of the vector instead of being constant $(= A)$ would vary as $A\varepsilon^{-\alpha t}$ or $A\varepsilon^{-\alpha x}$, and the full expression would be $A\varepsilon^{-\alpha t} \sin \omega t$ or $A\varepsilon^{-\alpha x} \sin \omega x$. The locus of the extremity of the vector would now be a centre-seeking spiral.

3. **Derivation of Network Equations.** The basic laws of the differential equations in electrical work, from which the unknown may be found in terms of the known quantities, are Kirchhoff's

laws. These are usually considered for steady-state conditions, but apply equally well to transient conditions. These laws are—

1. The sum of the e.m.f.s in any mesh or closed circuit is equal to the sum of the potential drops in that circuit.

2. The sum of the currents at any junction in a network is zero. Current flowing towards the junction is reckoned as positive, and that leaving is considered as negative.

There are also certain theorems which are of use in setting up the equations in certain instances, as follows—

(i) SUPERPOSITION THEOREM. The current that flows in a network, due to the simultaneous action of a number of e.m.f.s

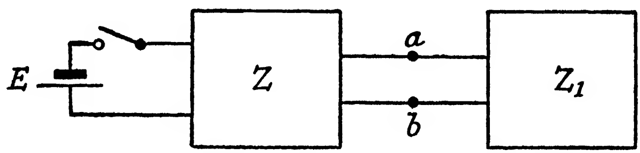

FIG. 1

applied in any manner, is the sum of the currents due to each of the component e.m.f.s acting separately.

(ii) RECIPROCITY THEOREM. If an e.m.f., located at a point in a circuit, produces a certain current at any other point in the circuit, then the same e.m.f. acting at the second point would produce the same current at the first point. In other words, the receiving point and the sending point may be interchanged.

(iii) THÉVENIN'S THEOREM.* The current in any circuit is equal to the open-circuit voltage divided by the sum of the external and internal impedances. For the circuit shown in Fig. 1, consisting of impedances Z and Z_1, where Z includes the impedance of the source, let the voltage across ab with Z_1 absent be e_t. Then the current with Z_1 present is given by

$$i = e_t/(Z + Z_1)$$

(iv) COMPENSATION THEOREM. If a network is modified by making a change ΔZ in the impedance of the network, the current increment ΔI, produced at any point in the network, is equal to the current that would be produced at that point by a compensating e.m.f. of value $I \,.\, \Delta Z$ acting in series with the modified branch, where I is the original current in the network.

* *Comptes rendus,* 1883, 97, 159.

4. **Differential Equations of a Network.** In all our subsequent discussions we assume that the current in every branch is directly proportional to the voltage, i.e. the parameters or coefficients of the equation are constants. In a circuit consisting of a resistance, inductance, and capacity in series we have that the voltage drop in the resistance is Ri by Ohm's law, in the inductance is $L \,.\, di/dt$ by Lenz's law, and in the capacity is $\int i/C \,.\, dt$. So that, if V is the applied voltage, we have

$$\begin{aligned} V &= Ri + L \,.\, di/dt + \int i/C \,.\, dt \\ &= (R + L \,.\, d/dt + \int 1/C \,.\, dt)i \\ &= Z_{11}i \end{aligned}$$

where Z_{11} denotes the *self-impedance* of the circuit. When circuits are coupled as shown in Fig. 2, there is a mutual effect

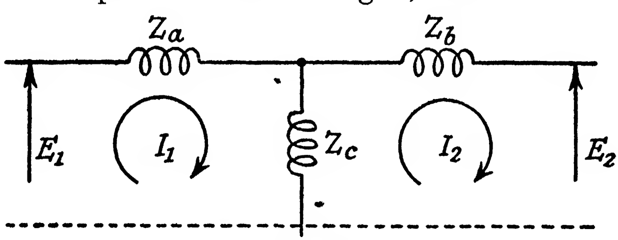

FIG. 2

between the currents flowing in the two circuits. For the current directions as shown, we have the following equations

$$\begin{aligned} I_1Z_a + I_1Z_c - I_2Z_c &= E_1 \\ I_2Z_b + I_2Z_c - I_1Z_c + E_2 &= 0 \end{aligned}$$

whence

$$I_1(Z_a + Z_c)\,(Z_b + Z_c) - I_1Z_c^2 = (Z_b + Z_c)E_1 - Z_cE_2$$

i.e.
$$I_1 = \frac{(Z_b + Z_c)E_1 - Z_cE_2}{(Z_a + Z_c)\,(Z_b + Z_c) - Z_c^2}$$

Denoting by Z_{11} the self-impedance of circuit 1, viz. $(Z_a + Z_c)$, by Z_{22} the self-impedance of circuit 2, viz. $(Z_c + Z_b)$, and by Z_{12} $(= Z_{21})$ the mutual impedance of circuit 1 with respect to circuit 2, viz. Z_c, we have, on substitution,

$$I_1 = \frac{Z_{22}E_1 - Z_{12}E_2}{Z_{11}Z_{22} - Z_{12}^2}$$

For a complex network we get

$$E_1 = Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 \dots + Z_{1n}I_n$$
$$E_2 = Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 \dots + Z_{2n}I_n$$

and so on.

From this array of n linear equations we obtain the equation for I_n in a concise form by use of determinants as follows.

Let D be the determinant of the impedance operators, viz.

$$D = \begin{vmatrix} Z_{11} & Z_{12} & \dots & Z_{1n} \\ Z_{21} & Z_{22} & \dots & Z_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ Z_{n1} & Z_{n2} & \dots & Z_{nn} \end{vmatrix}$$

Then the minor M_{ab} is formed from the above by deleting the ath column and the bth row.

We then obtain the following set of differential equations—

$$I_1 = E_1 \,.\, M_{11}/D + E_2 \,.\, M_{12}/D + \dots + E_n \,.\, M_{1n}/D$$
$$I_2 = E_1 \,.\, M_{21}/D + E_2 \,.\, M_{22}/D + \dots + E_n \,.\, M_{2n}/D$$

and so on.

The general differential equation is given by

$$I_r = \sum_{s=1}^{s=n} \cdot E_s \frac{M_{rs}}{D} \quad . \quad . \quad . \quad (1)$$

where $s = 1, 2, \dots n$.

5. **Heaviside Notation.** Following Heaviside, we introduce the symbol $p = d/dt$; the inverse operation of integration we denote by $1/p = \int_0^t (\quad)dt$. Prior to Heaviside, this notation had been used by both Caqué and Boole (1864–5). For repeated differentiation we have $p^n = d^n/dt^n$, and for repeated integration we have $1/p^n = \int_0^t \int_0^t \dots n$ times. This latter statement may be demonstrated by using Caqué's method of solution. Thus, suppose we have

$$Ri + Lpi = E$$

Then $\qquad pi = (E - Ri)/L$

or $\qquad i = (1/L \int_0^t Edt - (R/L) \int_0^t idt$

On integrating the first term and substituting the whole expression for i we obtain

$$i = Et/L - (R/L)\int\left[I/L\int Edt - (R/L)\int idt\right]$$

i.e.

$$i = \frac{E}{L}\cdot t - \frac{ER}{L^2}\cdot\frac{t^2}{2!} + \frac{R^2}{L^2}\int\int idt$$

Repeating this process, we get

$$i = \frac{E}{L}t - \frac{ER}{L^2}\cdot\frac{t^2}{2!} + \frac{ER^2}{L^3}\cdot\frac{t^3}{3!} - \ldots$$

i.e.

$$i = \frac{E}{L}\left[t - \frac{R}{2!L}\cdot t^2 + \frac{R^2}{3!L^2}\cdot t^3 \ldots\right] \quad . \quad (2)$$

or

$$i = \frac{E}{L}\cdot\frac{L}{R}\left(1 - \varepsilon^{-\alpha t}\right) = \frac{E}{R}\left(1 - \varepsilon^{-\alpha t}\right)$$

where $\alpha = R/L$.

Another method of dealing with this equation is what Heaviside calls *algebrizing*. On writing the original equation in symbolic form we get

$$i = \frac{E}{R + Lp}\cdot(1) = \frac{E}{Lp[1 + (R/L)(1/p)]}(1)$$

$$= \frac{E}{L}\left[\frac{1}{p} - \frac{R}{L}\cdot\frac{1}{p^2} + \frac{R^2}{L^2}\cdot\frac{1}{p^3}\ldots\right](1) \quad . \quad . \quad (3)$$

Comparing like terms in equations (2) and (3), we obtain that

$(1/p)$ is equivalent to* t

$$(1/p^2) \risingdotseq t^2/2!$$

and

$$(1/p^n) \risingdotseq t^n/n! \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Incidentally, we also obtain that

$$\frac{1}{p + \alpha}(1) \risingdotseq (1/\alpha)(1 - \varepsilon^{-\alpha t}) \quad . \quad . \quad . \quad (5)$$

Also, it may be demonstrated that the operator p obeys the usual laws of differentiation. Thus if A is a constant, and x, y variables with respect to time, we have

$$p(Ax) = Ap(x)$$

$$p(x + y) = p(x) + p(y)$$

$$p^n p^m(x) = p^{(n+m)}(x)$$

* The sign $\risingdotseq$ means "is equivalent to" wherever it occurs in the text.

A word of caution must be inserted for expressions like $\varepsilon^{-\alpha t} p\ (1)$, which is not equivalent to $p\varepsilon^{-\alpha t}\ (1)$. The p operates only on subsequent terms, and not on prior terms.

In the above discussion we have used a symbol introduced by Heaviside, namely (1). This denotes his *unit function.* He defined this function as having a zero value before $t = 0$ and a

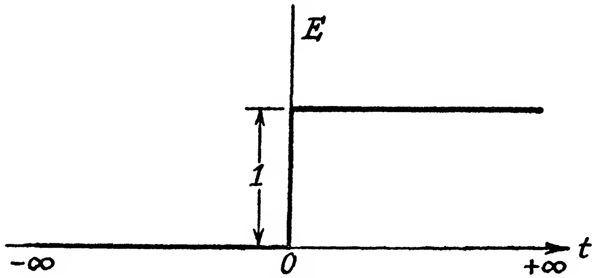

FIG. 3

constant value of unity thereafter (Fig. 3). So it represents a square wave of infinite period. For the beginner it is best to consider it as a battery of unit e.m.f. switched on to the line at $t = 0$. The mathematical basis of the idea will be dealt with later.

6. **Solution of the Differential Equation.** From Section 4 we have the differential equation

$$I_r = \sum_{s=1}^{s=m} E_s \frac{M_{rs}}{D}$$

Dropping the subscripts, and, in conformity with usual practice, writing $Y(p)$ and $Z(p)$ in place of M and D, we have

$$I = E \cdot \frac{Y(p)}{Z(p)}$$

Now $Z(p)$ will be a polynomial in p and may be written as

$$Z(p) = p^n + ap^{n-1} + bp^{n-2} + \dots$$

and $Y(p)$ will be a polynomial in p also, but of degree less than that of $Z(p)$. In view of the simplicity and straightforward nature of Heaviside's method of solution of such equations, we will proceed with his methods and neglect the formal mathematical methods which are obtainable elsewhere. Heaviside now proceeds as follows.

(i) By *algebrization* of the $Y(p)/Z(p)$ term in a power series. Frequently the result is difficult to convert to a known function.

This method has already been illustrated in Caqué's method of Section 5.

(ii) By the *expansion method*, described in his *Electromagnetic Theory*, Vol. II, p. 127, where he gives the following expansion

$$i = E\left[\frac{1}{Z_{(0)}} + \sum \frac{\varepsilon^{pt}}{p(dZ/dp)}\right]$$

"The Z in the operational solution is an operator, a function of p, the time differentiator. But Z is entirely algebraical. Thus $Z(0)$ is the algebraical function obtained by putting $p = 0$ in Z. Then, in the summation, dZ/dp is the ordinary differential coefficient of Z with respect to p as a quantity. Lastly, the summation ranges over all the roots of the algebraical equation $Z = 0$."

Heaviside merely sketched the proof of this expansion. However, several rigorous proofs now exist (see Bibliography), and we now give a method due to Goto* which is based on a method due to Press.†

The fundamental equation is given by

$$i = E[Y(p)/Z(p)]$$

By a well-known theorem on partial fractions‡ we have that

$$f'(x) = f(x)/(x - a) + f(x)/(x - b) + \ldots$$

where $f(x) = (x - a)(x - b) \ldots$

So with $x = a$ we get

$$[f'(x)]_{x=a} = (a - b)(a - c) \ldots = [f(x)/(x - a)]_{x=a}$$

Also

$$[f'(x)]_{x=b} = [f(x)/(x - b)]_{x=b}$$

As $Y(p)$ and $Z(p)$ are polynomials in p, we may write

$$Y(p)/Z(p) = A/(p - p_1) + B/(p - p_2) + \ldots \quad (6)$$

where $p_1, p_2, p_3, \ldots$ are the roots of $Z = 0$, and $A, B, C \ldots$ are constants. To determine these constants we proceed as follows.

Cross-multiplying equation (6), we obtain

$$Y(p) = A(p - p_2)(p - p_3) \ldots + B(p - p_1)(p - p_3) \ldots + C(p - p_1)(p - p_2) \ldots$$

* *Researches of Electrotechnical Laboratory*, Tokio, p. 201.

† *Trans. A.I.E.E.* (1917), Vol. 36, p. 232.

‡ WILLIAMSON, *Integral Calculus*, p. 49.

With $p = p_1$, we get

$$Y(p_1) = A(p_1 - p_2)(p_1 - p_3) \dots$$

or

$$A = \frac{Y(p_1)}{(p_1 - p_2)(p_1 - p_3) \dots}$$

By the preceding theorem

$$A = [Y/Z']_{p=p_1}$$

Similarly

$$B = [Y/Z']_{p=p_2}$$

So that

$$Y/Z = 1/(p - p_1) \, . \, [Y/Z']_{p=p_1} + 1/(p - p_2) \, . \, [Y/Z']_{p=p_2} + \dots$$

We consider the case when the applied voltage is given by $E\varepsilon^{\alpha t}$. Then

$$i = E\{1/(p - p_1) \, . \, [Y/Z']_{p=p_1} + 1/(p - p_2) \, . \, [Y/Z']_{p=p_2} + \dots\} \varepsilon^{\alpha t} \quad . \quad (7)$$

Now, by power expansion

$$1/(p - q) \, . \, \varepsilon^{\alpha t} = (1/p)(1 + q/p + q^2/p^2 + \dots)\varepsilon^{\alpha t}$$

Noting that

$$(1/p) \, . \, \varepsilon^{\alpha t} = \int_0^t \varepsilon^{\alpha t} dt = (1/\alpha)(\varepsilon^{\alpha t} - 1)$$

$$(1/p^2) \, . \, \varepsilon^{\alpha t} = \int_0^t (1/\alpha)(\varepsilon^{\alpha t} - 1) dt = (1/\alpha^2)(\varepsilon^{\alpha t} - 1) - (t/\alpha)$$

and generally

$$(1/p^n) \, . \, \varepsilon^{\alpha t} = (1/\alpha^n)\varepsilon^{\alpha t} - 1/\alpha^n - t/\alpha^{n-1} - t^2/2!\alpha^{n-2} \dots - t^{n-1}/(n-1)!\,\alpha$$

So that

$$\begin{aligned} 1/(p - q) \, . \, \varepsilon^{\alpha t} &= (1/p)\,[\varepsilon^{\alpha t}(1 + q/\alpha + q^2/\alpha^2 \dots) \\ &\quad - (q/\alpha)(1 + q/\alpha + q^2/\alpha^2 \dots) - (q^2 t/\alpha) \\ &\quad (1 + q/\alpha + q^2/\alpha^2 \dots) \dots] \\ &= (1/p)\,[(\varepsilon^{\alpha t} - q/\alpha - q^2 t/\alpha \dots) \, . \, 1/(1 - q/\alpha)] \\ &= (1/p)(\varepsilon^{\alpha t} - q\varepsilon^{qt}/\alpha) \, . \, 1/(1 - q/\alpha) \end{aligned}$$

Carrying out the integration, we get

$$1/(p - q) \, . \, \varepsilon^{\alpha t} = (\varepsilon^{\alpha t} - \varepsilon^{qt})/(\alpha - q)$$

Substituting this expression in equation (7), we get

$$i = E\,\{[1/(\alpha - p_1)]\,(\varepsilon^{\alpha t} - \varepsilon^{p_1 t})\,[Y/Z']_{p=p_1} + [1/(\alpha - p_2)]$$
$$(\varepsilon^{\alpha t} - \varepsilon^{p_2 t})\,[Y/Z']_{p=p_2} + \ldots\}$$

$$= E\varepsilon^{\alpha t}\,\{[1/(\alpha - p_1)]\,[Y/Z']_{p=p_1} + [1/(\alpha - p_2)]$$
$$[Y/Z']_{p=p_2} + \ldots\} - E\,\{[\varepsilon^{p_1 t}/(\alpha - p_1)]\,[Y/Z']_{p=p_1}$$
$$+ [\varepsilon^{p_2 t}/(\alpha - p_2)]\,[Y/Z']_{p=p_2} + \ldots\}$$

Now

$$[1/(\alpha - p_1)]\,[Y/Z']_{p=p_1} + [1/(\alpha - p_2)]\,[Y/Z']_{p=p_2} + \ldots = [Y/Z]_{p=\alpha}$$

So that we get

$$i = E\varepsilon^{\alpha t}[Y/Z]_{p=\alpha} - E\sum_{r=1}^{r=n}[\varepsilon^{p_r t}/(\alpha - p_r)]\,[Y/Z']_{p=p_r} \quad . \quad (8)$$

The value of α is determined by the form of the applied voltage. For a d.c. voltage, $\alpha = 0$, and then equation (8) becomes

$$i = E\,\{[Y/Z]_{p=0} + \Sigma\varepsilon^{p_n t}\,[Y/p_n Z']_{p=p_n}\} \quad . \quad . \quad . \quad (9)$$

which we will call the *first kind of expansion.*

For an a.c. voltage, $\alpha = j\omega$, and we have

$$i = E\left\{\varepsilon^{j\omega t}\left[\frac{Y}{Z}\right]_{p=j\omega} - \sum_{r=1}^{r=n}\frac{\varepsilon^{p_r t}}{j\omega - p_r}\left[\frac{Y}{Z'}\right]_{p=p_r}\right\} \quad . \quad . \quad (10)$$

which we call the *second kind of expansion.*

For a voltage given by $E \cos \omega t$, we obtain, on rationalizing and discarding imaginaries,

$$i = E\left\{\varepsilon^{j\omega t}\left[\frac{Y}{Z'}\right]_{p=j\omega} + \sum\frac{p_n}{p_n^2 + \omega^2}\,\varepsilon^{p_n t}\left[\frac{Y}{Z'}\right]_{p=p_n}\right\}\ (reals) \quad (11)$$

7. **Routine to be Adopted.** For a beginner with this method of solving equations it would be well to adhere to a fixed routine such as the following.

(i) Deduce the equation using the "p" notation. Suppose we get $p^2 i + api + bi = E\ .\ (1)$. This, according to our interpretation, means a constant voltage E is applied at $t = 0$, and is maintained indefinitely at that value.

(ii) By algebraic manipulation, we have

$$i = \frac{E}{p^2 + ap + b} \cdot (1)$$

Here note, $Y(p) = 1$ and $Z(p) = p^2 + ap + b$. On equating $Z(p)$ to zero we get $p = [-a \pm \sqrt{(a^2 - 4b)}]/2 = p_1 p_2$.

(iii) By differentiation, we have $Z'(p) = 2p + a$.

(iv) For $p = 0$, we get $Z(0) = b$ and $Y(0) = 1$.

(v) By collecting terms and substituting in the expansion expression

$$i = E\,\{[Y/Z]_{p=0} + \Sigma[(Y/pZ')\varepsilon^{pt}]_{p=p_n}\}$$

we get

$$i = E[1/b + \varepsilon^{p_1 t}/p_1(2p_1 + a) + \varepsilon^{p_2 t}/p_2(2p_2 + \alpha)]$$

(vi) On substituting the values for p_1 and p_2, the complete solution of the differential equation is obtained. It will be noted that by this method no integration constants are left to be evaluated. This is because of the assumption that at $t = 0$ the circuit is dead. This, as will be seen later, does not hold for all cases, but by a simple extension we are able to bring our solution into line with any circuit conditions. In Chapter II we will follow this routine in solving certain circuit problems.

8. **Discussion on Roots of** $Z(p) = 0$. The solution of the equation by means of the expansion theorem depends on our ability to extract the roots of $Z(p) = 0$. A little consideration will show the difficulties we are liable to encounter in analytical work, but, fortunately, in a numerical example, we are able to approximate to the roots by either Horner's or Graeffe's method. These are explained fully in many texts.*

If the roots obtained from $Z(p) = 0$ are all different, then the application of the expansion theorem is straightforward. But when certain of the roots are equal, then Wagner† shows that for roots $p_1, p_2 \ldots p_n$ and p_m, the latter occurring m times, we have

$$i = E\left\{\frac{Y(0)}{Z(0)} + \sum_{r=1}^{r=n}\left[\frac{Y(p_r)}{p_r Z'(p_r)}\,\varepsilon^{pt}\right]_{p=p} + \varepsilon^{p_m t}(B_1 + B_2 t + \ldots B_m t^{m-1})\right\}$$

* E.g. Whittaker and Robinson: *Calculus of Observations* (Blackie).
† *Elect. für Elek.*, 1916, 159.

where B_1, B_2 . . . are constants determined by terminal conditions.

Also, it should be noted that the proof of the expansion theorem is dependent on $Y(p)$ being of degree less than that of $Z(p)$. If this condition is not fulfilled, we must divide through until our remainder $R(p)$ is of power less than that of $Z(p)$ as

$$Y(p)/Z(p) = B_1p^2 + B_2p + \ldots + R(p)/Z(p)$$

Frequently it is possible to get out of this difficulty by solving for i/p, say, and then differentiating the result to obtain i.

PRODUCTS. Sometimes the impedance operator $Z(p)$ is given as the product of two components, $Z_1(p) \cdot Z_2(p)$. To evaluate such an expression, recourse is had to the superposition theorem of Borel (see p. 56), which may be written operationally as follows—

$$1/Z(p) \doteqdot A(t) = d/dt \int_0^t A_1(t-\lambda) \cdot A_2(\lambda) d\lambda$$

$$= A_1(t) \cdot A_2(0) + \int_0^t A_1(t-\lambda) \cdot A_2'(\lambda) d\lambda$$

where $A_1(t) = 1/Z_1(p)$ and $A_2(t) = 1/Z_2(p)$.

We will illustrate this important theorem by an example. We have already shown in equation (5) that

$$\frac{1}{p+\alpha} \cdot (1) \doteqdot \frac{1}{\alpha}[1 - \varepsilon^{-\alpha t}]$$

Then by differentiation $\dfrac{p}{p+\alpha} \cdot (1) \doteqdot \varepsilon^{-\alpha t}$

Suppose now that we are given

$$\frac{1}{Z(p)} \doteqdot \frac{\alpha}{p+\alpha} \cdot \frac{p}{p+\alpha}$$

Then we have

$$\frac{1}{Z_1(p)} \doteqdot \frac{\alpha}{p+\alpha}(1) \doteqdot (1 - \varepsilon^{-\alpha t}) = A_1(t)$$

and $$\frac{1}{Z_2(p)} = \frac{p}{p+\alpha}(1) \doteqdot \varepsilon^{-\alpha t} \qquad = A_2(t)$$

By substitution we get

$$A(t) = [1 - \varepsilon^{-\alpha t}] + \int_0^t [1 - \varepsilon^{-\alpha(t-\lambda)}][-\alpha\varepsilon^{-\alpha\lambda}] d\lambda$$

On integrating, we get

$$A(t) = [1 - \varepsilon^{-\alpha t}] + \alpha t\varepsilon^{-\alpha t} + \varepsilon^{-\alpha t} - 1$$
$$= \alpha t\varepsilon^{-\alpha t}$$

HIGHLY OSCILLATORY CIRCUIT. Guillemin* shows that for a low dissipation circuit we may split our determinantal equation into two parts, and, by combining the roots so obtained, get a close approximation to the actual roots. For a full discussion of this method we refer the reader to the original, but the working may be illustrated by the following example taken from the original—

Let $$Z(p) = p^4 + 50p^3 + 8{\cdot}91 \times 10^5p^2 + 2{\cdot}03 \times 10^7p + 10^{11}$$

Now let $D(p) = p^4 + 8{\cdot}91 \times 10^5p^2 + 10^{11} = 0$

This gives $p = \pm j800$, or $p = \pm j500$.

Also, let $G(p) = 50p^3 + 2{\cdot}03 \times 10^7p$

On substituting the above values of p, we have

$$G(p) = \pm j9{\cdot}36 \times 10^9, \text{ or } G(p) = \pm j3{\cdot}90 \times 10^9$$

Then

$$d/dp\,.\,[D(p)] = 4p^3 + 17{\cdot}82 \times 10^5p$$

Let $$\delta = -\frac{G(p)}{D'(p)}$$

Then $$\delta = -\frac{j9{\cdot}36 \times 10^9}{j6{\cdot}23 \times 10^8} = -15$$

or $$\delta = -\frac{j3{\cdot}90 \times 10^9}{j3{\cdot}90 \times 10^8} = -10$$

So that the approximate roots $(\delta + p)$ are $-15 \pm j800$ and $-10 \pm j500$.

9. **Notes on Roots.** Frequently in discussions of circuit conditions the prime consideration is stability, and so we need not solve our equations, for such information (i.e. regarding stability) may be obtained from the determinantal equation. The underlying theory can be obtained from the theory of equations, and we give below a brief summary of some of the rules involved.

* *Trans. I.E.E.* (1928), Vol. 47, p. 361.

General. 1. For $f(x) = 0$, there cannot be more positive roots than there are changes in sign of $f(x)$. Also, there cannot be more negative roots than there are changes in sign of $f(-x)$.

Thus $$f(x) = x^9 + 5x^8 - x^3 + 7x + 2 = 0$$

Here there are two changes, so we may expect two positive roots.

Also $$f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2 = 0$$

Now, there are three changes in sign, and so we may expect three negative roots. Altogether there will be nine roots, and so the other four must be imaginary.

2. If all coefficients are real, then the imaginary roots must occur in conjugate pairs.

Quadratic Equations. For the equation $ax^2 + bx + c = 0$, the roots are given by the well-known expression

$$x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$$

If all coefficients are positive, then the roots are negative. For roots to be positive, the coefficient b must be negative by general rule 1.

1. For real roots, $b^2 > 4ac$.
2. For equal roots, $b^2 = 4ac$.
3. For imaginary roots, $b^2 < 4ac$.

As these will be of frequent occurrence in the text, we refer the reader to later discussion.

Cubic Equations. Here the general equation is of the form—

$$ax^3 + bx^2 + cx + d = 0$$

On substituting $(-x)$ for x, we find, in accordance with Rule 1, that there are three changes in sign, and so there may be three negative roots. Usually in solving these equations $[x - (b/3a)]$ is substituted for x, and an equation of the form $x^3 + qx + r = 0$ results.

From this equation we are able to state—

(a) If $r^2/4 + q^3/27$ is positive, then the roots are of the form

$$\alpha + \beta;\ -\frac{(\alpha + \beta)}{2} + \frac{(\alpha - \beta)}{2}\sqrt{-3};\ -\frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2}\sqrt{-3}$$

(b) If $r^2/4 + q^3/27$ is zero, the roots are 2α; $-\alpha$; $-\alpha$.

(c) If $r^2/4 + q^3/27$ is negative, then α^3 and β^3 are imaginary quantities. So let $\alpha = \gamma + j\delta$, $\beta = \gamma - j\delta$.

Then the roots are 2γ; $-\gamma - \delta\sqrt{3}$; $-\gamma + \delta\sqrt{3}$, i.e. real

quantities. The conditions for non-oscillation may be shown to be $b^2 - 3ac > 0$ and $b^2c^2 - 4ac^3 > 27a^2d^2 - 18\,abcd + 4b^3d$.

Biquadratic Equations. Let us consider an equation of the form

$$m^4 + C_1m^3 + C_2m^2 + C_3m + C_4 = 0$$

The roots of this equation, if C_1, C_2, C_3, C_4 are real and positive, will be real or complex conjugate. They will occur in pairs, two reals and two complex. Now, it may be shown that if p_1, p_2, p_3, p_4, are the roots, then

$$(p_1 + p_2)(p_1 + p_3)(p_1 + p_4)(p_2 + p_3)(p_2 + p_4)$$
$$= C_1C_2C_3 - C_4C_2^2 - C_3^2 = X \text{ (say)}$$

(a) If the roots are $a_1 \pm jb_1$ and $a_2 \pm jb_2$, we obtain

$$2(a_1 + a_2) = -C_1 \quad . \quad . \quad . \quad (12)$$

and

$$X = 4a_1a_2[(a_1 + a_2)^2 + (b_1 + b_2)^2][(a_1 + a_2)^2 + (b_1 - b_2)^2] \quad (13)$$

From equation (12), and for a stable damped system, a_1, a_2 must be negative, and so we have X positive.

(b) If the roots are $a_1 \pm jb_1$ and $a_2 \pm b_2$, then we have

$$2(a_1 + a_2) = -C \quad . \quad . \quad . \quad (14)$$

and $$X = 4a_1a_2\ \{[(a_1 + a_2)^2 + b_1^2 - b_2^2]^2 + 4b_1^2b_2^2\} \quad . \quad (15)$$

Here we have the same criterion, viz. if a_1, a_2 are both negative, then X is positive.

(c) If the roots are $p_1 \pm jq_1$; $r_1 \pm js_1$, then we have the following relationships between roots and coefficients—

$$p_1 + r_1 = -C_1/2$$
$$p_1^2 + q_1^2 + r_1^2 + s_1^2 + 4p_1r_1 = C_2$$
$$r_1(p_1^2 + q_1^2) + p_1(r_1^2 + s_1^2) = -C_3/2$$
$$(p_1^2 + q_1^2)(r_1^2 + s_1^2) = C_4$$

If we make p_1 zero, we then get jq_1 and $r_1 \pm js_1$, so that

$$r_1 = -C_1/2$$
$$q_1^2 + r_1^2 + s_1^2 = C_2$$
$$r_1q_1^2 = -C_3/2$$
$$q_1^2(r_1^2 + s_1^2) = C_4$$

Whence we get $$q_1^2(C_2 - q_1^2) = C_4$$

or $$-\frac{C_3}{2r_1}\left(C_2 + \frac{C_3}{2r_1}\right) = C_4$$

$$\therefore\ C_1C_2C_3 - C_3^2 - C_1^2C_4 = 0 \quad . \quad . \quad (16)$$

This is an important relationship in systems, since for stability p_1 and r_1 must be negative. Now increase p_1 to, say, $p_1 = 0$. Then equation (16) gives the limiting characteristics for stability.

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CHAPTER II

APPLICATION OF EXPANSION THEOREM TO LUMPED CIRCUITS

(A) Making the Circuit with D.C. Applied Volts

1. WE now apply the expansion theorem to simple lumped circuits, i.e. with concentrated characteristics. We concentrate on the problem of the making of the circuit by the application of a d.c. battery, of voltage E, to the circuit. At the time $t < 0$, the circuit is dead, and at time $t = 0$ a voltage E is applied. We are supposing that an ideal switch to give instantaneous application of voltage, without any leakage current, is available.

2. **The RL Circuit.** For a circuit consisting of resistance R and inductance L, at any instant we have the following equation—

$$Ri + Lpi = E$$

For a suddenly applied voltage, we write, in accord with previous discussion—

$$Ri + Lpi = E(1)$$

So that

$$i = E\frac{1}{R + Lp}(1) = E\frac{Y(p)}{Z(p)} \cdot (1)$$

where $Z(p) = R + Lp$ and $Y(p) = 1$.

Following our routine, we get—

$$R + Lp = 0; \quad \therefore p = -R/L$$
$$Z'(p) = L$$
$$Z(0) = R,\ Y(0) = 1$$

So, on substituting in the expansion theorem, we have—

$$i = E\left[\frac{1}{R} + \frac{1}{L(-R/L)}\varepsilon^{-Rt/L}\right] = (E/R)[1 - \varepsilon^{-Rt/L}] \quad . \quad (17)$$

where ε is the base of Napierian logarithms.

As t approaches infinity, the current approaches E/R. This is the steady-state current, and is usually denoted by I_0. Our equation becomes—

$$i = I_0[1 - \varepsilon^{-\alpha t}]$$

where $\alpha = R/L$.

3. Deductions from Above Equation. (*a*) On differentiating with respect to t, we obtain

$$di/dt = \alpha I_0 \varepsilon^{-\alpha t}$$

and at $t = 0$ we get $\quad di/dt = \alpha I_0$

The initial rate of growth is proportional to α, so a coil of large resistance and small inductance will reach its steady state more quickly than a coil of small resistance and large inductance. After a lapse of time $1/\alpha = L/R$, the current is $I_0[1 - 1/\varepsilon]$, i.e. $0{\cdot}623 I_0$. This value of time is known as the *time constant* of the circuit.

(*b*) The voltage drop and power consumption in the components of the circuit are dealt with as follows. Across the resistance the voltage drop is Ri, or $E[1 - \varepsilon^{-\alpha t}]$, and across the inductance it is $L(di/dt)$, or $E\varepsilon^{-\alpha t}$. By adding these components we see that the sum of the voltage drops is equal to the applied voltage at any instant.

The energy dissipated as heat at any instant is

$$Ri^2 = (E^2/R)[1 - 2\varepsilon^{-\alpha t} + \varepsilon^{-2\alpha t}]$$

and the energy stored in the magnetic field at any instant is

$$iL(di/dt) = (E^2/R)\varepsilon^{-\alpha t}[1 - \varepsilon^{-\alpha t}]$$

The power consumed between $t = 0$ and $t = T$ is given by

$$\int_0^T Ri^2 dt + \int_0^T Li(di/dt)dt \text{ watts}$$

Substituting the value of i and simplifying, we get—

$$(E^2/R)[T + (1/\alpha)\varepsilon^{-\alpha T} - (1/\alpha)]$$

When T is large, this expression becomes $(E^2/R)[T - 1/\alpha)$ and as a limiting case it becomes E^2T/R.

4. The Condenser Circuit [CR]. In considering the charging of a condenser we pass over the interesting case of charging a perfect condenser and consider the case of a condenser C with a resistance R in the circuit. Initially, let the charge on the condenser be zero, so that the voltage becomes

$$Ri + (1/pC)i = E(1)$$

i.e. $$i = E \cdot \frac{1}{R + (1/Cp)} \cdot (1) = E\,\frac{Cp}{RCp + 1} \cdot (1)$$

Here $Y(p) = Cp$, and $Z(p) = RCp + 1$, so $Y(0) = 0$, $Z(0) = 1$, and $Z'(p) = RC$.

From $Z(p) = 0$ we have $p = -1/CR$. From the expansion theorem we get

$$i = E\left[0 + \frac{C(-1/CR)\varepsilon^{t/CR}}{RC(-1/RC)}\right] = (E/R)\,\varepsilon^{-t/CR} \quad . \quad (18)$$

The charge $q = \int_0^t idt = EC[1 - \varepsilon^{-\alpha t}]$, where $\alpha = 1/CR$.

The values of the time constant, the voltage across the circuit components, the energy, and the power consumption may be determined in the same way as shown above in Section 3.

5. **The** RLC **Series Circuit.** For a resistance, inductance, and capacity in series, the equation becomes

$$Ri + Lpi + (1/pC)i = E\,(1)$$

so that $i = \dfrac{E}{R + Lp + (1/pC)}\,(1) = E\,\dfrac{Cp}{LCp^2 + RCp + 1}\,(1)$

Now, $Y(p) = Cp$ and $Z(p) = LCp^2 + RCp + 1$

also $Y(0) = 0$ and $Z(0) = 1$

The roots obtained from $Z(p) = 0$ are

$$p = -\frac{R}{2L} \pm \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}$$

It is usual practice to denote $R/2L$ by α and $1/LC$ by ω_0^2, where ω_0 is the natural angular velocity. Also $\sqrt{\left(\dfrac{R^2}{4L^2} - \dfrac{1}{LC}\right)}$ is denoted by ω_1. Then $\omega_1^2 = \alpha^2 - \omega_0^2$. Thus the roots are $p = -\alpha \pm \omega_1$.

By differentiation we get

$$Z'(p) = [2LCp + RC]_{p_1, p_2} = \pm\sqrt{(R^2C^2 - 4LC)} = \pm 2LC\omega_1$$

On substituting in the expansion formula we get

$$i = EC\left[0 + \frac{p_1\,\varepsilon^{p_1 t}}{p_1\,.\,2LC\omega_1} - \frac{p_2\,\varepsilon^{p_2 t}}{p_2\,.\,2LC\omega_1}\right]$$
$$= (E/2L\omega_1)\,[\varepsilon^{p_1 t} - \varepsilon^{p_2 t}]$$

Inserting the values of p_1 and p_2, we get

$$i = (E/2\omega_1 L)\,\varepsilon^{-\alpha t}[\varepsilon^{\omega_1 t} - \varepsilon^{-\omega_1 t}] \quad . \quad . \quad (19)$$

Further development depends on the value of ω_1. From Section 9, Chapter I, we see that there are three cases—

(i) If ω_1^2 is positive, i.e. $R^2/4L^2 > 1/LC$, we get

$$i = (E/\omega_1 L)\,\varepsilon^{-\alpha t}\sinh\omega_1 t \quad . \quad . \quad (20)$$

The value of i approaches its final value slowly, and the condition is known as *overdamped.*

(ii) If ω_1^2 is negative, i.e. $1/LC > R^2/4L^2$, then we must substitute $j\omega_1$ for ω_1 in the above equation, and so obtain

$$i = (E/\omega_1 L)\,\varepsilon^{-\alpha t}\sin\omega_1 t \qquad . \qquad . \quad (21)$$

In this case the current is oscillatory, its period is equal to the free period of the circuit, and the circuit is said to be *underdamped.*

(iii) If $\omega_1 = 0$, i.e. $R^2/4L^2 = 1/LC$, then $\sinh\omega_1 t$ approaches $\omega_1 t$, and thus the limit of $(\sinh\omega_1 t)/\omega_1$ is t.

This gives

$$i = (E/L)t\,\varepsilon^{-\alpha t} \qquad . \qquad . \qquad . \quad (22)$$

The current here approaches its final value without oscillation and at a faster rate than under (i). This condition is known as *critical damping*, and the value of R obtained from

$$R^2/4L^2 = 1/LC$$

is $2\sqrt{(L/C)}$ and is known as the *critical resistance* of the circuit.

(B) Making the Circuit with A.C. Applied Volts

6. **The RL Circuit.** Since an alternating voltage may be represented as the real part of $E\varepsilon^{j\omega t} = E[\cos\omega t + j\sin\omega t]$, we now apply equation (10) of Section 6, Chapter I, namely—

$$i = E\left[\frac{Y(j\omega)}{Z(j\omega)}\varepsilon^{j\omega t} - \sum_{r=1}^{r=n}\frac{Y(p)\,\varepsilon^{p_r t}}{(j\omega - p_r)Z'(p_r)}_{p=p_r}\right]$$

The values of $Y(p)$, $Z(p)$, $Z'(p)$, and the roots are identical with those obtained in Section 2, Chapter II; so by substitution in the above formula we get

$$i = E\left\{\frac{1}{R + j\omega L}\varepsilon^{j\omega t} - \frac{\varepsilon^{-(R/L)t}}{[j\omega + (R/L)]L}\right\}$$

$$= E\left[\frac{\cos\omega t + j\sin\omega t}{R + j\omega L} - \frac{\varepsilon^{-(R/L)t}}{R + j\omega L}\right]$$

Rationalizing and discarding imaginaries from the above equation, we have

$$i = E\left[\frac{R\cos\omega t + \omega L\sin\omega t}{R^2 + \omega^2 L^2} - \frac{R}{R^2 + \omega^2 L^2}\varepsilon^{-(R/L)t}\right]$$

Letting $\tan\phi = \omega L/R$, we get

$$i = [E/\sqrt{(R^2 + \omega^2 L^2)}]\,[\cos(\omega t - \phi) - \cos\phi\,.\,\varepsilon^{-(R/L)t}] \quad . \quad (23)$$

The steady-state current will be $(E/Z)\cos(\omega t-\phi)$. It will be noted that the time constant of the circuit is not affected by the type of voltage applied to the circuit.

We next consider a voltage $E\varepsilon^{j(\omega t+\theta)}$ applied to the circuit, i.e. the voltage is $E\varepsilon^{j\theta}$ when the time is zero. Then we have

$$i = E\left[\frac{\varepsilon^{j(\omega t+\theta)}}{R+j\omega L}-\frac{\varepsilon^{j\theta}\,\varepsilon^{-(R/L)t}}{R+j\omega L}\right]$$

Discarding imaginaries, we get

$$i = [E/\sqrt{(R^2+\omega^2L^2)}]\,[\cos(\omega t+\theta-\phi)-\cos(\theta-\phi)\,.\,\varepsilon^{-(R/L)t}] \qquad (24)$$

When $\theta=0$ it is seen that this expression reduces to that given above. The transient term depends on the value of the angle θ. When $\theta=\phi$, we get the maximum transient current.

We may regard the transient term as raising or lowering the time axis. On this distorted axis the steady-state condition is superimposed. So, if we produce the resultant steady-state current back to meet the initial current, it will be found that sometimes they meet without any phase distortion but at others do not do so. In the former case there would be no transient.

7. **The *CR* Circuit.** The voltage equation may be written as

$$Rpq+q/C=E\varepsilon^{j(\omega t+\theta)}\,.\,(1)$$

Completion of the solution along the lines of the last paragraph gives

$$q=[EC/\sqrt{(1+\omega^2R^2C^2)}]\,[\cos(\omega t+\theta-\phi)-\cos(\theta-\phi)\,.\,\varepsilon^{-t/CR}]$$

where $\tan\phi=\omega CR$. By differentiation we get

$$i=-[EC/\sqrt{(1+\omega^2R^2C^2)}]\,[\omega\sin(\omega t+\theta-\phi)-(1/CR)\cos(\theta-\phi)\varepsilon^{-t/CR}] \qquad (25)$$

8. **The *RLC* Circuit.** For an applied voltage $E\varepsilon^{j\omega t}$ we get

$$i[R+Lp+(1/pC)]=E\varepsilon^{j\omega t}(1)$$

or

$$i=\frac{E}{R+Lp+(1/pC)}\varepsilon^{j\omega t}(1)$$

Adopting the notation of Section 5, Chapter II, we have $p=-\alpha\pm\omega_1$ and $Z'(p)=\pm 2LC\omega_1$, and on substituting we get

$$i=EC\left[\frac{j\omega\,\varepsilon^{j\omega t}}{-\omega^2LC+j\omega RC+1}-\sum_1^2\frac{p_n\varepsilon^{p_nt}}{\pm(j\omega-p_n)\,.\,2LC\omega_1}\right]$$

On simplifying and discarding imaginaries, we have that

$$i = EC\left[\frac{\omega\cos(\omega t + \phi)}{\sqrt{[\omega^2R^2C^2 + (1 - \omega^2LC)^2]}} + \frac{\varepsilon^{-\alpha t}}{\omega_1 LC}\cdot\right.$$

$$\left.\frac{[(\alpha^2 - \omega_1{}^2)^2 + \omega^2(\alpha^2 + \omega_1{}^2)]\sinh\omega_1 t - 2\alpha\omega_1\omega^2\cosh\omega_1 t}{(\alpha^2 - \omega_1{}^2 - \omega^2)^2 + 4\omega^2\alpha^2}\right] \quad (26)$$

where $\tan\phi = (1 - \omega^2LC)/\omega RC$ . . (26A)

The development of this equation depends on the relative values of $R^2/4L^2$ and $1/LC$.

(i) When $R^2/4L^2 > 1/LC$, then $\omega_1{}^2$ is positive and we are not able to simplify the above expression.

(ii) When $R^2/4L^2 < 1/LC$, then $\omega_1{}^2$ is negative and we have

$$i = E\left\{\frac{\cos(\omega t + \phi)}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\right.$$

$$\left. - \frac{\varepsilon^{-\alpha t}}{\omega_1 L}\cdot\frac{(\alpha^2 + \omega_1{}^2)\cos(\omega_1 t + \psi)}{\sqrt{[(\alpha^2 + \omega_1{}^2 - \omega^2)^2 + 4\alpha^2\omega^2]}}\right\} \quad . \quad (27)$$

where $\tan\psi = [(\alpha^2 + \omega_1{}^2)^2 + \omega^2(\alpha^2 - \omega_1{}^2)]/2\alpha\omega_1\omega^2$. (27A)

(iii) When $\omega_1 = 0$, the circuit is critically damped. We get, as ω_1 approaches 0, that $(\sinh\omega_1 t)/\omega_1$ approaches t and $\cosh\omega_1 t$ approaches 1, so equation (26) becomes

$$i = E\left\{\frac{\cos(\omega t + \phi)}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\right.$$

$$\left. + \frac{\varepsilon^{-\alpha t}}{L}\left[\frac{\alpha^2}{\alpha^2 + \omega^2}t - \frac{2\alpha\omega^2}{(\alpha^2 + \omega^2)^2}\right]\right\} \quad . \quad (28)$$

It is of interest to see the effects of impressed oscillations. Substituting $\alpha = R/2L$ and $\omega_0{}^2 = 1/LC$, equation (27) becomes

$$i = \frac{E}{L}\left\{\frac{\omega\cos(\omega t + \phi)}{\sqrt{[4\omega^2\alpha^2 + (\omega_0{}^2 - \omega^2)^2]}}\right\} - A\varepsilon^{-\alpha t}\cos(\omega_1 t + \psi)$$

where A is a constant.

It is evident that—

(*a*) The system gives up its natural period (ω_1) and assumes that of the impressed force (ω) with lapse of time.

(*b*) When the impressed and natural frequencies are almost equal and the damping coefficient α is small, the two components will assist each other during one interval and neutralize during the next. The system would thus produce a series of beats."

(*c*) When $\omega = \omega_1$ and α/ω is very small, then $\omega_0{}^2 - \omega^2 = \alpha^2$

and $\tan\phi = \alpha/2\omega$, which approaches zero. The amplitude may be written as

$$i = \frac{E\omega}{L\sqrt{[4\omega^2\alpha^2 + (\omega_0{}^2 - \omega^2)^2]}} = \frac{E}{L\sqrt{(4\omega^2\alpha^2 + \alpha^4)}}$$

$$= \frac{E\omega}{\alpha L\sqrt{(4\omega^2 + \alpha^2)}} = \frac{E\omega}{2\alpha L\omega\sqrt{(1 + \alpha^2/4\omega^2)}} \cong \frac{E}{R}$$

which is very large.

9. **Numerical and Logarithmic Decrement.** Using the expression for current as given in Section 5, Chapter II, viz.

$$i = (E/\omega_1 L)\,\varepsilon^{-\alpha t}\sin\omega_1 t,$$

and differentiating, we get

$$di/dt = (E/\omega_1 L)\,\varepsilon^{-\alpha t}\,[\omega_1\cos\omega_1 t - \alpha\sin\omega_1 t]$$

On equating $di/dt = 0$, we obtain that $\tan\omega_1 t = (\omega_1/\alpha)$ or generally $\tan(\omega_1 t + 2n\pi) = (\omega_1/\alpha)$ where n is an integer. So at $t_n = [\tan^{-1}(\omega_1/\alpha) - 2n\pi]/\omega_1$ the current is a maximum. By substituting this value of time, we get that

$$i_{max} = (E/\omega_1 L)\varepsilon^{-\alpha t_n}\sin[\tan^{-1}(\omega_1/\alpha) - 2n\pi]$$

The first maximum will be obtained when $n = 0$—

$$i_{1\,max} = (E/\omega_1 L)\varepsilon^{-\alpha t_n}\sin\tan^{-1}(\omega_1/\alpha) = [E/L\sqrt{(\alpha^2 + \omega_1{}^2)}]\varepsilon^{\alpha}$$

For $n = 1$ we get the second maximum as

$$i_{2\,max} = [E/L\sqrt{(\alpha^2 + \omega_1{}^2)}]\varepsilon^{-\alpha t_1}$$

Or generally we get

$$i_{n\,max} = [E/L\sqrt{(\alpha^2 + \omega_1{}^2)}]\varepsilon^{-\alpha t_n}$$

The interval between successive maxima is $2\pi/\omega_1$ and the difference of the currents is

$$i_n - i_{n+1} = [E/L\sqrt{(\alpha^2 + \omega_1{}^2)}]\,[\varepsilon^{-\alpha t_n} - \varepsilon^{-\alpha t_{n+1}}]$$

$$= [E/L\sqrt{(\alpha^2 + \omega_1{}^2)}]\,[1 - \varepsilon^{\alpha(2\pi/\omega_1)}]\cdot\varepsilon^{-\alpha t_n}$$

or $$\frac{i_n - i_{n+1}}{i_n} = 1 - \varepsilon^{+\alpha 2(\pi/\omega_1)} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This expression is known as the *numerical decrement.*

For the *logarithmic decrement* we take

$$\log_e(i_n/i_{n+1}) = \log_e.\,\varepsilon^{\alpha(2\pi/\omega_1)} = \alpha(2\pi/\omega_1)$$

$$= R/2Lf_1 = \delta\ \text{(say)} \quad . \quad . \quad (30)$$

where $f_1 = \omega_1/2\pi$.

10. **Natural Period and Critical Resistances.** We have defined the natural period of the RLC circuit as

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

So for an undamped circuit we obtain

$$f_0 = (1/2\pi)\sqrt{(1/LC)}$$

and thus we get
$$f_1^2 = f_0^2 - \frac{1}{4\pi^2}\left(\frac{R^2}{4L^2}\right)$$
$$= f_0^2 - \delta^2 f_1^2/4\pi^2$$
$$\therefore f_0/f_1 = \sqrt{(1 + \delta^2/4\pi^2)} \qquad . \qquad . \ (31)$$

This latter formula is well known to radio engineers.

We have seen that the value of the resistance which makes the term under the root sign vanish is given by $2\sqrt{(L/C)}$. This is the critical resistance of the circuit.

If the value of the applied frequency in an RLC circuit given in equation (26) be altered, then the maximum steady-state current is obtained when $\omega^2R^2C^2 + (1 - \omega^2LC)^2$ is a minimum, i.e. when $2\omega R^2C^2 + 4\omega^3L^2C^2 - 4\omega LC = 0$.

So we get a value of

$$R = \pm\sqrt{[(2L/C) - 2\omega^2L^2]} = \pm L\sqrt{2}\sqrt{[(1 - \omega^2LC)/LC]} \ (32)$$

These may be defined as *frequency critical resistances.*

11. **General Tuning or Resonance Conditions.** The equation for forced periodic motion may be written as

$$ap^2\theta + bp\theta + c\theta = F\cos\omega t$$

Then, in accord with Section 8, we obtain that the steady state deflection is given by

$$\theta = F\cos(\omega t - \phi)/\sqrt{[\omega^2b^2 + (c - a\omega^2)^2]}$$

where $\cot\phi = (c - a\omega^2)/\omega b$.

Let the impressed frequency be continuously varied. Then the amplitude of θ will be a maximum when

$$\omega^2 = (c/a) - \tfrac{1}{2}(b/a)^2 = w_r^2 \text{ (say)} = \text{resonant angular velocity.}$$

As the natural angular velocity of the system is given by $\omega_0^2 = c/a$, we have $w_r^2 = \omega_0^2 - \frac{1}{2}(b/a)^2$.

By substitution, the amplitude is given by

$$2aF/b\sqrt{(4ac - b^2)} = F/b\omega_n$$

where $\omega_n^2 = (4ac - b^2)/4a^2 = \omega_0^2 - \frac{1}{4}(b/a)^2$.

Now, the critical value of $b = b_c = 2\sqrt{(ca)}$, so we have the amplitude of the deflection as

$$\theta = F/c\sqrt{\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + 4\,\frac{b^2\omega^2}{b_c^2\omega_0^2}\right]}$$

Let $\gamma = \omega/\omega_0$ and $\delta = b/b_c$. We then have

$$\theta = F/c\sqrt{[(1-\gamma^2)^2 + 4\delta^2\gamma^2]} \quad . \quad . \quad (33)$$

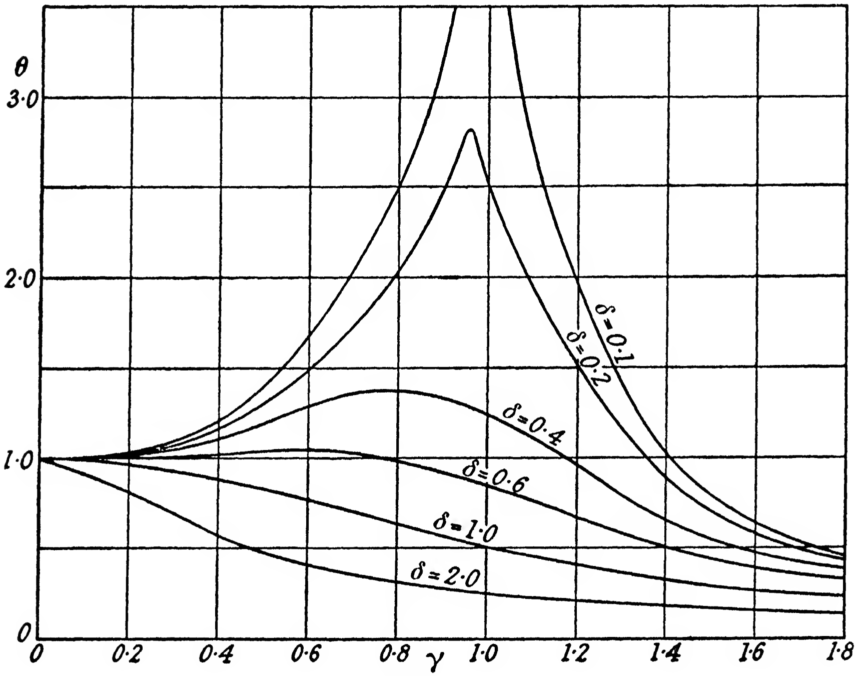

Fig. 4

When $\omega \to 0$, i.e. applied frequency approaches zero, we get $\theta_0 = F/C$, so that

$$\frac{\text{amplitude}}{\text{amplitude at zero frequency}} = \frac{\theta}{\theta_0} = \frac{1}{\sqrt{[(1-\gamma^2)^2 + 4\delta^2\gamma^2]}} \quad . \quad (34)$$

Also $$\tan\phi = 2\delta\gamma/(1-\gamma^2) \quad . \quad . \quad . \quad (35)$$

As these results are very general, we have plotted the results in Figs. 4 and 5 for various degrees of damping (δ) against γ. It is seen that the peak of the oscillations tends toward $\omega/\omega_0 = 1$ as the damping is reduced. Inspection of the phase-angle curves of Fig. 5 shows that for low frequencies the resulting motion and deflecting forces are almost in phase. But as the

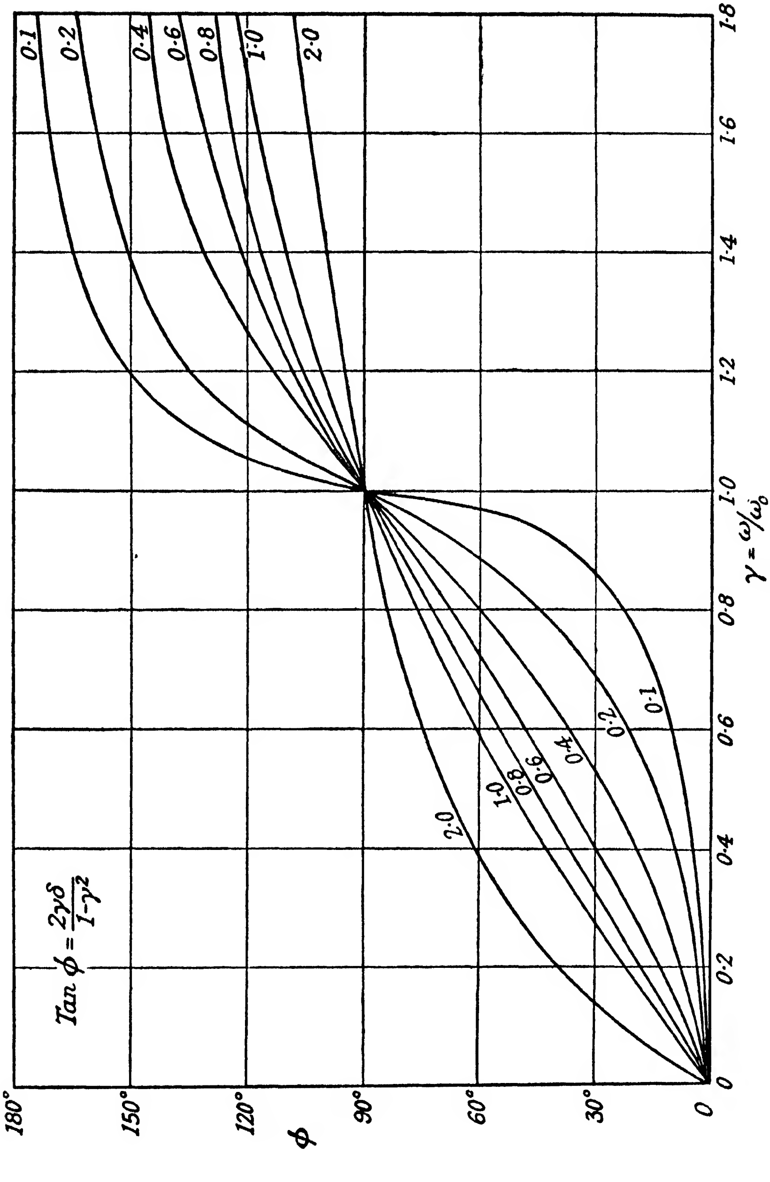
$Tan\,\phi = \frac{2\gamma\delta}{1-\gamma^2}$
ϕ
180°
150°
120°
90°
60°
30°
0
0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
$\gamma = \omega/\omega_0$
0·1
0·2
0·4
0·6
0·8
1·0
2·0

FIG. 5

frequency is increased the motion lags more and more behind the deflecting force. At $\omega/\omega_0 = 1$ they are in quadrature.

12. **Plotting of Results.** In analytical work mathematicians have made much use of non-dimensional quantities. We use this idea for plotting the results of equations such as have been given in previous sections. By proper selection of a time base we are able to use one curve for all problems of a type. Thus for the LR circuit we have

$$Ri + Lpi = E$$

If $t = (L/R)x$, then $dt = (L/R)dx$. Then, on substituting, we get

$$Ri + L\frac{dx}{dt}\frac{di}{dx} = Ri + R\Delta i = E$$

where $\Delta = d/dx$ or

$$i = \frac{E}{R(1+\Delta)}(1) = (E/R)[1 - \varepsilon^{-x}] \qquad . \quad (36)$$

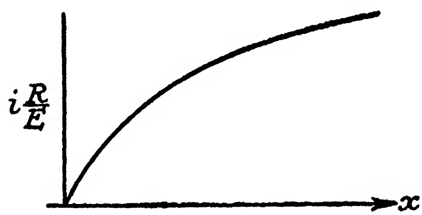

FIG. 6

Thus for such a circuit we only need to plot the curve $(1 - \varepsilon^{-x})$ against x, as in Fig. 6. Such a curve we will designate as the *master curve.* To use it on any circuit, all that is necessary is to multiply the ordinates by E/R and the value of x by L/R.

When this idea is applied to equations of the second power, we have

$$p^2y + (\alpha + \beta)py + \alpha\beta y = E$$

or

$$y = \frac{E}{(p+\alpha)(p+\beta)} \cdot (1)$$

By partial fractions we get

$$y = \frac{E}{\beta - \alpha}\left[\frac{1}{p+\alpha} - \frac{1}{p+\beta}\right] \cdot (1)$$

Thus, considering an RLC circuit as

$$p^2q + (R/L)pq + (1/LC)q = E/L$$

we now have $\alpha, \beta = \dfrac{R}{2L} \mp \sqrt{\left(\dfrac{R^2}{4L^2} - \dfrac{1}{LC}\right)}$

In the above equation q has two components, q_1 and q_2. Letting $e = (E/L)/(\beta - \alpha)$, we have

$$q_1 = \frac{1}{p+\alpha}\left[\frac{E}{L}\cdot\frac{1}{\beta-\alpha}\right] = e\cdot\frac{1}{p+\alpha}$$

and $$q_2 \qquad = e\cdot\frac{1}{p+\beta}$$

So that by carrying out the method illustrated for a first power equation we get that

$$\alpha t = x_1$$

$$q_1 = \frac{e}{\alpha}\left[\frac{1}{\Delta + 1}\right]\cdot(1)$$

$$= \frac{e}{\alpha}[1 - \varepsilon^{-x}] \quad . \quad . \quad . \quad . \quad (37)$$

and $$q_2 = \frac{e}{\beta}[1 - \varepsilon^{-y}]$$

where $\beta t = y_1$. So that

$$(\beta - \alpha)q = \frac{E}{L\alpha}(1 - \varepsilon^{-x}) - \frac{E}{L\beta}(1 - \varepsilon^{-y}) \quad . \quad (38)$$

Thus we now plot $(1 - \varepsilon^{-x})$ and $(1 - \varepsilon^{-y})$ and multiply the ordinates by $E/L\alpha(\beta - \alpha)$ and $E/L\beta(\beta - \alpha)$ respectively. By subtracting ordinates at like times, we obtain the values of q.

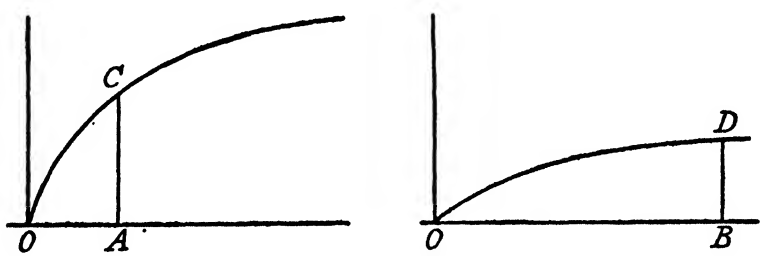

FIG. 7

Thus in Fig. 7 let OA and OB be identical times. Then the value of $q = AC - BD$. Curves of this nature are useful where a considerable amount of checking is necessary. In order to plot the master curves, all that is required is a set of tables of exponential functions.

Practical Examples

We will now discuss certain applications which may be solve by the matter already developed. Actually we have use electrical circuits in our discussion, but we will show that th principles hold for other subjects as well.

13. **Sweep Circuit for Cathode-ray Oscilloscope.** For ou first example we take that of obtaining a variable time bas or sweep, in the cathode-ray oscillograph. In this instrumen the current to be examined is applied to one set of coils; t another set of coils, at right angles to the former set, a "sweep current is applied. This current causes the beam to be swep across the viewing screen, and thus produces a wave form One method is to use a circuit containing a reactance. From previous work we know that the current grows according to

$$i = (E/R)[1 - \varepsilon^{-\alpha t}]$$

$$= (E/R)\,[\alpha t - \alpha^2 t^2/2! + \ldots] \quad . \quad . \quad (39)$$

As the time is here of the order of microseconds, it is seen tha

$$i \cong (E/R)\alpha t = Et/L \quad . \quad . \quad . \quad (40)$$

i.e. the sweep deflecting force varies inversely as L.

For a particle of charge e moving at a speed v, the equivalen coil $= ev$. If B is the flux density set up by the sweep magne coil, the force on a particle is evB per unit length. But *force* $=$ *mass* $\times$ *acceleration*, and $l = \frac{1}{2}at^2$.

$$\therefore\ l = \tfrac{1}{2}at^2 = \tfrac{1}{2}(evB/m)t^2$$

and $$t = \sqrt{(2lm/evB)} = iL/E$$

or $$l = (L/E)^2 i^2 (evB/2m) \text{ cm.} \quad . \quad . \quad . \quad (41)$$

where l is the width of the screen swept out.

Electrical		Mechanical	
Name	Symbol	Symbol	Name
Time	t	t	Time
Charge	q	δ	Deflection
Inductance	L	M	Mass
Current	qt^{-1}	δt^{-1}	Velocity
Voltage	qLt^{-2}	$M\delta t^{-2}$	Force
Power	q^2Lt^{-3}	$M\delta^2 t^{-3}$	Power
Resistance	Lt^{-1}	Mt^{-1}	Damping
Capacity	$L^{-1}t^2$	$M^{-1}t^2$	Resilience

14. **Motion of Indicating Instruments.** For allied mechanical problems we are able to make use of the *principle of duality.* The table given on page 28 has been drawn up showing corresponding quantities in electrical and mechanical units. We assume that inductance and mass and also charge and deflection are corresponding quantities. From a table such as this, once the differential equation has been set up, we are able to discuss the problem fully by referring to the corresponding electrical equation.

In the case of an instrument movement, a force F acts on the moving coil. This force is set up by the interaction of the current in the coil and the field of the permanent magnet. It will be resisted by a force due to inertia, a force due to the damping vanes, and by elastic forces set up in the instrument springs. For small movements we may take the damping force as proportional to the velocity, and the shear or control force as proportional to the deflection.

If Δ is the deflection, then $d\Delta/dt = p\Delta$ is the velocity and $p^2\Delta$ the acceleration. Now, the force equation may be written

$$mp^2\Delta + \rho p\Delta + \tau\Delta = F$$

where ρ = resisting force per unit velocity and τ = control force per unit deflection.

$$\therefore \Delta = \frac{F}{mp^2 + \rho p + \tau} \cdot (1)$$

for an impulse force.

Now, $Z(p) = mp^2 + \rho p + \tau$ and $Z(0) = \tau$, and the roots are given by

$$p = -\frac{\rho}{2m} \pm \sqrt{\left(\frac{\rho^2}{4m^2} - \frac{\tau}{m}\right)} = -\alpha \pm \beta$$

$$Z'(p) = 2mp + \rho = \pm 2m\beta$$

so that $$\Delta = F\left[\frac{1}{\tau} + \frac{1}{2m\beta}\left(\frac{\varepsilon^{p_1 t}}{p_1} - \frac{\varepsilon^{p_2 t}}{p_2}\right)\right]$$

Inserting the values of p_1 and p_2, we obtain that

$$\Delta = F\left\{\frac{1}{\tau} + \frac{\varepsilon^{-\alpha t}}{2m\beta(\alpha^2 - \beta^2)}\right.$$
$$\left.[\alpha(\varepsilon^{\beta t} - \varepsilon^{-\beta t}) + \beta(\varepsilon^{\beta t} + \varepsilon^{-\beta t})]\right\} \quad (42)$$

As for the corresponding electrical circuit, further development depends on the value of β.

(i) If β is positive, then the above equation gives the final result. The instrument is said to be *overdamped*, and the pointer moves slowly to its final reading without overshooting.

(ii) If β is zero, then

$$\Delta = F\left(\frac{1}{\tau} + \frac{t\varepsilon^{-\alpha t}}{2m\alpha^2}\right) \quad . \quad . \quad . \quad (43)$$

The instrument approaches its final reading more quickly than in case (i), and is now said to be *critically damped.* Usually the inherent damping is not sufficient to produce this result, and external resistance is incorporated.

(iii) When β^2 is negative, then the root is of the form $\pm j\beta$, and we obtain

$$\Delta = F\left[\frac{1}{\tau} - \frac{\varepsilon^{-\alpha t}}{m\beta\sqrt{(\alpha^2 + \beta^2)}}\cos(\beta t - \phi)\right] \quad . \quad (44)$$

where $\tan\phi = \alpha/\beta$.

Here the movement is oscillatory, the pointer swinging about its final reading. In this condition the instrument is said to be *underdamped.*

15. **Oscillograph Vibrators.** The general theory of oscillographs may be obtained elsewhere. Here we will be concerned with the interesting case of obtaining a correct trace, i.e. the relative amplitudes and phase relationships of the various harmonics to the fundamental must be preserved. The disturbing force varies harmonically and is given by $F \sin \omega t$ for the fundamental, so in our notation the deflection due to the fundamental is given by

$$\Delta = \frac{F \sin \omega t}{mp^2 + \rho p + \tau} \cdot (1)$$

By the expansion theorem we obtain that

$$\Delta = F\left\{\frac{\varepsilon^{j\omega t}}{\tau - \omega^2 m + j\omega\rho} - \frac{\varepsilon^{-\alpha t}}{2m\beta(j\omega - p_1)(j\omega - p_2)}\right.$$
$$\left.[(j\omega - p_2)\varepsilon^{\beta t} - (j\omega - p_1)\varepsilon^{-\beta t}]\right\}$$

where p_1 and p_2 are the roots of $mp^2 + \rho p + \tau = 0$, viz. $-\alpha \pm \beta$.

For recurrent phenomena we may allow the vibrator to remain in circuit for some time without registering, and so we are interested in the particular integral only, viz.—

$$\Delta = F \cdot \frac{\varepsilon^{j\omega t}}{\tau - m\omega^2 + j\omega\rho}$$

On rationalizing we obtain

$$\Delta = F \cdot \frac{\sin(\omega t - \phi)}{\sqrt{[\omega^2\rho^2 + (\tau - m\omega^2)^2]}} \quad . \quad . \quad (45)$$

where $\tan\phi = \rho\omega/(\tau - m\omega^2)$.

In oscillographic work the disturbing force will be of the form $c\sum_{n=1}^{n=\infty} I_n \sin(n\omega t - \beta_n)$, where c is a constant converting the current into a force and n is the number of the harmonic. Under this condition the integral becomes

$$\Delta = c\sum_{n=1}^{n=\infty}\left\{\frac{I_n \sin(n\omega t - \beta_n - \phi_n)}{\sqrt{[\rho^2 n^2\omega^2 + (\tau - mn^2\omega^2)^2]}}\right\} \quad . \quad (46)$$

and $\tan\phi_n = \rho n\omega/(\tau - mn^2\omega^2)$.

Inspection shows that if m and ρ were zero, the trace would be an exact facsimile of the wave. As this condition is impossible of attainment in practice, the damping of the instrument should be as near critical as possible, or $\rho^2 = 4m\tau$.

Substituting, we have

$$\Delta_1 = c\sum_{n=1}^{n=\infty}\frac{I_n \sin(n\omega t - \beta_n - \phi_n')}{\tau + n^2\omega^2 m} \quad . \quad . \quad (47)$$

where $\tan\phi_n' = 2n\omega\sqrt{(\tau m)}/(\tau - n^2\omega^2 m)$.

But when $m = \rho = 0$, the deflection is

$$\Delta_0 = c\sum_{n=1}^{n=\infty}\frac{I_n \sin(n\omega t - \beta_n)}{\tau} \quad . \quad . \quad (48)$$

The difference between Δ_1 and Δ_0 gives the error existing in the resultant trace. The ratio of the amplitudes is

$$\frac{\Delta_1}{\Delta_0} = \frac{\tau}{\tau + n^2\omega^2 m} = \frac{1}{1 + (n^2\omega^2 m/\tau)} = R_n, \text{ say} \quad . \quad (49)$$

The natural or free period of the instrument is given by

$$T_0 = 2\pi\sqrt{(m/\tau)}$$

Also $n^2\omega^2/4\pi = n^2 f^2$, where f is the fundamental frequency of the phenomena under investigation. Let $1/f = T = $ the period, then

$$R_n = \frac{1}{1 + n^2(T_0/T)^2} \quad . \quad . \quad . \quad (50)$$

As this ratio is less than unity, there will be an error depending on the harmonic—the higher the harmonic, the greater the error in the amplitude. Also the phase angle now becomes

$$\tan \phi_n = \frac{\rho n\omega}{\tau - n^2\omega^2 m} = \frac{2n}{(T/T_0) - n^2(T_0/T)} \quad . \quad (51)$$

and it is seen that there is a small phase shift among the components.

In the recording of transient phenomena we cannot allow the time delay necessary for the decay of the transient deflection, so it is essential that free vibrations are non-existent or at least damped out as rapidly as possible. As the critically damped or aperiodic instrument gives the best results under steady-state conditions, we will consider the response of this instrument to transient conditions. In general, the force due to the action of current in the vibrator will be represented as

$$F = F_0 + \sum_{n=1}^{n=\infty} F_n \varepsilon^{-\mu t} \sin n\omega t$$

and on substituting this in our force equation we get

$$\Delta = A\varepsilon^{-Kt} \sin\left[\sqrt{\left(\omega_1^2 - \frac{K^2}{4}\right)}t + \alpha\right] + \frac{F_0}{\omega_1^2}$$
$$+ \sum \frac{F_n \varepsilon^{-\mu t} \sin(n\omega t - \phi_n)}{\sqrt{[\omega_1^2 - n^2\omega^2 + \mu(\mu - K)]^2 + n^2\omega^2(2\mu - K)^2}}$$

where $\tan \phi_n = \dfrac{n\omega(K - 2\mu)}{\omega_1^2 - n^2\omega^2 + \mu(\mu - K)}$

and $\qquad \omega_1^2 = \tau/m, \; K = \rho/2m \quad . \quad . \quad . \quad (52)$

The first term of the above solution represents free oscillations, which will die away unless $K \leq 0$. Here we require K to be large, or the value of $\omega^2 - K^2/4 = 0$, i.e. aperiodic motion. So then we have that the first term will be given by

$$\Delta_T = \frac{F}{m}\left[\frac{\omega}{\omega^2 + (\beta - \alpha)^2} \frac{\varepsilon^{(-\alpha+\beta)t}}{2\beta} - \frac{\omega}{\omega^2 + (-\beta - \alpha)^2} \frac{\varepsilon^{(-\alpha-\beta)t}}{2\beta}\right]$$

When β approaches zero, we have

$$\Delta_T \to \frac{F}{m} \cdot \frac{\omega}{\omega^2 + \alpha^2} t\varepsilon^{(-\beta)t} \quad . \quad . \quad . \quad (53)$$

16. **To Increase the Range of an Oscillograph.** As an oscillograph requires to be critically damped to give a true record

over a range of frequencies, the range of frequencies is restricted to the natural period of the vibrator. An interesting method of increasing the range of an instrument by means of a resonant shunt has been devised by Irwin* (Oscillograph) and Wynn Williams.† The circuit is as shown in Fig. 8.

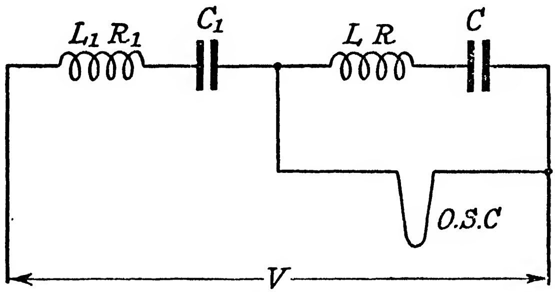

FIG. 8

Now, as the current through the oscillograph is infinitesimal, we may write

$$[(L_1 + L)p^2 + (R_1 + R)p + 1/C' + 1/C]q = V$$

as the voltage across the whole circuit.

Across the oscillograph strip we get

$$[Lp^2 + Rp + 1/C]q = e$$

But e is proportional to the oscillograph deflection x, and so we write that

$$[Lp^2 + Rp + 1/C]q = x$$

Thus we obtain

$$x = \frac{V(Lp^2 + Rp + 1/C)}{(L_1 + L)p^2 + (R_1 + R)p + 1/C_1 + 1/C}$$

Let L_1 be negligible and write $\alpha = R/2L$, $\alpha_1 = R_1/2L$, $\omega_0^2 = 1/LC$, and $\omega_1^2 = 1/LC_1$.

We have $$x = \frac{V(p^2 + 2\alpha p + \omega_0^2)}{p^2 + 2(\alpha + \alpha_1)p + (\omega_0^2 + \omega_1^2)} \cdot (1) \quad . \quad (54)$$

From this equation we see that the damping is increased to $(\alpha + \alpha_1)$ and the natural period to $\sqrt{(\omega_0^2 + \omega_1^2)}$. If we assume $\alpha_1 = 4\alpha$ and $\omega_1^2 = 24\omega_0^2$, then it is seen that the damping coefficient becomes 5α and the natural period $= 5\omega_0$. Thus we would obtain an amplitude scale which is the same as if the galvanometer had a period of $5\omega_0$ and a damping constant of 5α.

* *Oscillographs.* Pitman 1925.

† *Phil. Mag.* (1925), Vol. 50, p. 1

CHAPTER III

SERIES AND PARALLEL CIRCUITS

1. **General Solutions.** The operational notation leads to neater expressions when we are dealing with series and parallel circuits. To illustrate the method we will consider RL circuits.

(i) For the series arrangement we have that

$$i[R_1 + L_1 p] = e_1$$

and
$$i[R_2 + L_2 p] = e_2$$

On adding these expressions, we have

$$E = e_1 + e_2 = i[R_1 + R_2 + (L_1 + L_2)p]$$

$$\therefore \; i = [E/(R_1 + R_2)]\,[1 - \varepsilon^{-(R_1 + R_2)t/(L_1 + L_2)}] \quad . \quad . \quad (55)$$

The time constant is given by $(L_1 + L_2)/(R_1 + R_2)$.

(ii) For the parallel arrangement we have

$$(R_1 + L_1 p)i_1 = E$$

and
$$(R_2 + L_2 p)i_2 = E$$

But
$$i_1 + i_2 = I$$

$$\therefore \; I = (E/R_1)\,[1 - \varepsilon^{-(R_1 t/L_1)}] + (E/R_2)\,[1 - \varepsilon^{-(R_2 t/L_2)}]$$

$$= E[(R_1 + R_2)/R_1 R_2 - (1/R_1)\varepsilon^{-(R_1 t/L_1)} - (1/R_2)\,\varepsilon^{-(R_2 t/L_2)}] \quad . \quad (56)$$

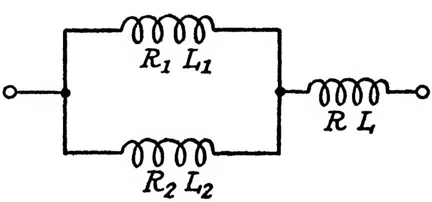

FIG. 9

As the general method is obvious, we consider now certain practical applications.

2. **High-speed Circuit-breaker.** To reduce the damage due to "flashing over" of rotary converters on electric traction systems, a circuit-breaker with a high speed of operation has been developed. For details of construction reference should be made to McNairy.* A simplified diagram of connections is shown in Fig. 9, where R_1, L_1 are the trip-coil constants, R_2, L_2 those of the holding-in coil, and R, L those of the load.

* McNAIRY: *Trans. A.I.E.E.* (1926), Vol. 45, p. 962.

We now have that

$$i_1[R_1 + L_1p] = i_2[R_2 + L_2p] \quad . \quad . \quad (57)$$

and also

$$i_1 + i_2 = I \quad . \quad . \quad . \quad (58)$$

As we may neglect the impedances of the coils in comparison with that of the load under fault conditions, we get

$$I = \frac{E}{R + Lp} \cdot (1)$$

From equations (57) and (58) we get

$$i_1[R_1 + R_2 + (L_1 + L_2)p] = [R_2 + L_2p]I$$

or

$$i_1 = \frac{I(R_2 + L_2p)}{R_1 + R_2 + (L_1 + L_2)p} \cdot (1)$$

$$= \frac{E}{R + Lp} \cdot \frac{R_2 + L_2p}{R_1 + R_2 + (L_1 + L_2)p} \cdot (1)$$

We now use Borel's theorem (page 56) to evaluate this operator. Denoting R/L by α and $(R_1 + R_2)/(L_1 + L_2)$ by β we get

$$L(L_1 + L_2)i_1 = E\left[\frac{R_2}{(p + \alpha)(p + \beta)} + \frac{L_2p}{(p + \alpha)(p + \beta)}\right] \cdot (1) \quad . \quad (59)$$

But

$$[1/(p + \alpha)] \cdot (1) \doteqdot (1/\alpha)[1 - \varepsilon^{-\alpha t}],$$

and

$$[p/(p + \alpha)] \cdot (1) \doteqdot \varepsilon^{-\alpha t} \text{ (see page 11)}$$

So the expression for the admittance is

$$A(t) = A_1(t) \cdot A_2(0) + \int_0^t A_1(t - \lambda) \cdot A_2'(\lambda) \cdot d\lambda$$

For the first term of equation (59) it becomes

$$A(t) = 1/(p + 1/\alpha)(p + \beta)$$

$$= \int_0^t A_1(t - \lambda) \cdot A_2'(\lambda) d\lambda$$

$$= \int_0^t (1/\alpha)[1 - \varepsilon^{-\alpha(t - \lambda)}]\varepsilon^{-\beta\lambda} d\lambda$$

On integrating we get

$$(1/\alpha)[(1 - \varepsilon^{-\beta t})/\beta - (\varepsilon^{-\beta t} - \varepsilon^{-\alpha t})/(\alpha - \beta)]$$

For the second term of equation (59) we get

$$\begin{aligned} A(t) &= p/(p+\alpha)(p+\beta) \\ &= \int_0^t \varepsilon^{-\alpha(t-\lambda)}\varepsilon^{-\beta\lambda}\,d\lambda \\ &= (\varepsilon^{-\beta t} - \varepsilon^{-\alpha t})/(\alpha - \beta) \end{aligned}$$

Combining the results, we have

$$\begin{aligned} i_1 &= \frac{E}{L[L_1+L_2]}\left[\frac{R_2}{\alpha}\left(\frac{1-\varepsilon^{\beta t}}{\beta} - \frac{\varepsilon^{-\beta t}-\varepsilon^{-\alpha t}}{\alpha-\beta}\right) + \frac{L_2}{\alpha-\beta}(\varepsilon^{-\beta t}-\varepsilon^{-\alpha t})\right] \\ &= \frac{E}{L_1+L_2}\left[\frac{R_2}{R}\cdot\frac{1-\varepsilon^{-\beta t}}{\beta} - \left(\frac{R_2}{R}-\frac{L_2}{L}\right)\left(\frac{\varepsilon^{-\beta t}-\varepsilon^{-\alpha t}}{\alpha-\beta}\right)\right] \quad . \quad (60) \end{aligned}$$

and
$$\frac{di_1}{dt} = \frac{E}{L_1+L_2}\left[\frac{R_2}{R}\cdot\varepsilon^{-\beta t} - \left(\frac{R_2}{R}-\frac{L_2}{L}\right)\left(\frac{\alpha\varepsilon^{-\alpha t}-\beta\varepsilon^{-\beta t}}{\alpha-\beta}\right)\right] \quad . \quad (61)$$

Now, i_1 is the trip-coil current, and from equation (61) the rate of initial rise is given by

$$\left[\frac{di_1}{dt}\right]_{t\to 0} = EL_2/L(L_1+L_2)$$

The current will attain its maximum when $di_1/dt = 0$; and after further working we get

$$\varepsilon^{(\alpha-\beta)t} = \alpha(R_2L - L_2R)/(\alpha R_2L - \beta L_2R)$$

So that the time lapse for the trip-coil current to attain its maximum is given by

$$t = [1/(\alpha-\beta)\,.\,\text{lgn.}\,[\alpha(R_2L - L_2R)/(\alpha R_2L - \beta L_2R)] \quad (62)$$

3. **Impulse Generator.** In routine testing of insulators, it is usual practice to apply an impulse voltage of wave shape approximating to that of a lightning surge. The circuit adopted is that due to Marx.* It consists of a bank of condensers which are connected in parallel for charging, and are discharged in series. A bank consisting of forty-eight condensers, each

* MARX: *E.T.Z.* (1924).

charged to 75 kV, would give 3 600 kV on discharge. Due to internal impedance, a pressure of the order of 3 000 kV would be available. An elementary connection diagram is shown in Fig. 10, where C_1 is the lumped generator capacity and C_2 the capacity of the insulator under test and the capacity to ground of the whole equipment.

Let the voltage across the insulator be e, and the voltage on the condenser C_1 be E, at $t = 0$. Also, let the current

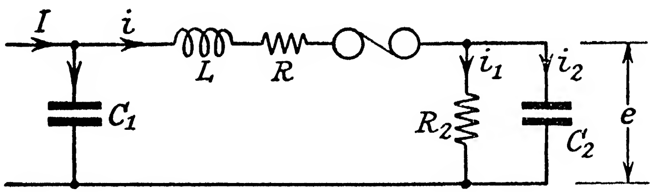

FIG. 10

distribution be as shown in the diagram. Let the volts across the condenser C_1 be e_c.

Then $\quad e_c = (R + Lp)i + e$

Also $\quad e_c = (1/C_1p) . (I - i)$ and $(1/p)I = Q$

Thus $\quad (R + Lp)i + e = (1/C_1p) . (I - i)$

or $\quad (R + Lp + 1/C_1p)i + e = Q/C_1 = E$

But $\quad i = i_1 + i_2 = e[(1 + R_2C_2p)/R_2] . (1)$

so that $$e = E \frac{R_2/(1 + R_2C_2p)}{R + Lp + 1/C_1p + R_2/(1 + R_2C_2p)} (1) \quad . (63)$$

This expression may be put in the following form—

$$e = E . \frac{p/LC_2}{p^3 + (R/L + 1/R_2C_2)p^2 + (R/LC_2R_2 + 1/LC_1 + 1/LC_2)p + 1/LC_1C_2R_2} . (1) \quad (64)$$

$$= E . \frac{Ap}{p^3 + ap^2 + bp + c} (1)$$

where $A = 1/LC_2$,

$a = (R/L + 1/R_2C_2)$,

$b = R/LC_2R_2 + 1/LC_1 + 1/LC_2$

and $\quad c = 1/LC_1C_2R_2$

As these coefficients are all positive, there can be no positive roots. For a third-degree equation, one root will be real and negative, and the other two imaginary and negative. So we may write equation (64) as

$$e = E\,\frac{Ap}{(p+\alpha)\,[(p+\lambda)^2+\omega^2]}\,.\,(1)$$

Factorizing, we get

$$e = \frac{AE}{(\lambda-\alpha)^2+\omega^2}\left[\frac{p}{p+\alpha} - \frac{\lambda-\alpha}{\omega}\cdot\frac{\omega p}{(p+\lambda)^2+\omega^2} - \frac{p^2+\lambda p}{(p+\lambda)^2+\omega^2}\right].\,(1)$$

By the expansion theorem we have (see Appendix)

$$e = \frac{AE}{(\lambda-\alpha)^2+\omega^2}\left[\varepsilon^{-\alpha t} - \frac{\lambda-\alpha}{\omega}\,\varepsilon^{-\lambda t}\sin\omega t + \varepsilon^{-\lambda t}\cos\omega t\right] \quad . \quad (65)$$

The last two terms are oscillatory and should be eliminated, i.e. ω^2 must be 0 or negative.

With $\omega = 0$ we have

$$e = \frac{AE}{(\lambda-\alpha)^2+\omega^2}\,[\varepsilon^{-\alpha t} - (\lambda-\alpha)t\varepsilon^{-\lambda t} + \varepsilon^{-\lambda t}] \quad . \quad (65\text{A})$$

As the value of R_2 is very large, we may consider the circuit for all practical purposes as RLC_1C_2, and from Chapter II we have that the critical resistance is given by

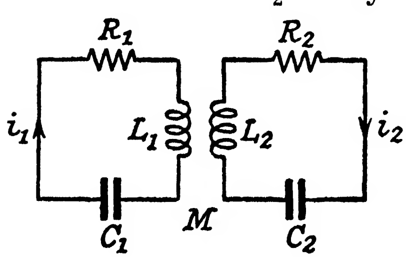

FIG. 11

$$R = 2[\sqrt{L(1/C_1 + 1/C_2)}]$$

This value of resistance must not be exceeded, otherwise the discharge will be oscillatory. In practice a wave is inspected by means of an oscillograph for ripples, and these should not exceed $\pm$ 5 per cent of the maximum amplitude of the wave.

4. **Mutually Coupled Circuits in Free Oscillation.** Systems may be so coupled that a disturbance in one member is reflected into a second member. The reflection occurs electromagnetically as in a transformer, or mechanically as in the

case of a turbo-alternator (Chapter VI). Let us consider the case shown in Fig. 11. At a time $t = 0$ let an impulse oscillation be set up in the primary circuit. Then we have the voltage equations as

$$L_1pi_1 + R_1i_1 + (1/C_1p)i_1 + Mpi_2 = 0 \quad . \quad (66)$$

$$L_2pi_2 + R_2i_2 + (1/C_2p)i_2 + Mpi_1 = 0 \quad . \quad (67)$$

Eliminating i_2 between these equations, we get

$$i_1[(L_1L_2 - M^2)p^4 + (R_1L_2 + R_2L_1)p^3 + (L_1/C_2 + R_1R_2 + L_2/C_1)p^2 + (R_1/C_2 + R_2/C_1)p + 1/C_1C_2] = 0$$

Adopting the notation $\alpha_1 = R_1/2L_1$, $\alpha_2 = R_2/2L_2$, $\omega_1{}^2 = 1/L_1C_1$, $\omega_2{}^2 = 1/L_2C_2$, and $K = M/\sqrt{(L_1L_2)}$ we get

$$L_1L_2i_1[(1 - K^2)p^4 + 2(\alpha_1 + \alpha_2)p^3 + (\omega_1{}^2 + 4\alpha\alpha_2 + \omega_2{}^2)p^2 + 2(\alpha_1\omega_2{}^2 + \alpha_2\omega_1{}^2)p + \omega_1{}^2\omega_2{}^2] = 0 \quad (68)$$

and an identical equation for i_2.

The determinantal equation is of the form

$$ap^4 + bp^3 + cp^2 + dp + f = 0 \quad . \quad . \quad (69)$$

For a highly oscillatory circuit with low dissipation, the roots may be extracted by the method of Guillemin (Chapter I) as follows—

$$D(p) = ap^4 + cp^2 + f$$

and so

$$p^2 = [- c \pm \sqrt{(c^2 - 4af)}]/2a$$

$$G(p) = bp^3 + dp$$

$$- \delta = (bp^2 + d)/(4ap^2 + 2c)$$

The roots are given by

$$\frac{bc \mp b\sqrt{(c^2 - 4af)} - 2ad}{\pm 2\sqrt{(c^2 - 4af)}} \pm j\sqrt{\left[\frac{c \mp \sqrt{(c^2 - 4af)}}{2a}\right]} \quad . \quad (70)$$

As the roots are conjugate, denote them by $-\gamma_1 \pm j\beta_1$ and $-\gamma_2 \pm j\beta_2$. Now, with conjugate roots the integration constants are also conjugate, so let them be given in the form $A\varepsilon^{+j\lambda}$ and $B\varepsilon^{+j\delta}$. The solution is thus in the form

$$i_1 = A\varepsilon^{-\gamma_1 t}[\varepsilon^{j(\beta_1 t + \lambda)} + \varepsilon^{-j(\beta_1 t + \lambda)}] + B\varepsilon^{-\gamma_2 t}[\varepsilon^{j(\beta_2 t + \delta)} + \varepsilon^{-j(\beta_2 t + \delta)}]$$

$$= A_1\varepsilon^{-\gamma_1 t}\cos(\beta_1 t + \lambda) + B_1\varepsilon^{-\gamma_2 t}\cos(\beta_2 t + \delta) \quad . \quad (71)$$

For the natural angular velocity we have that

$$\beta^2 = [c \mp \sqrt{(c^2 - 4af)}]/2a$$
$$= \frac{(\omega_1{}^2+4\alpha_1\alpha_2+\omega_2{}^2)\mp\sqrt{[(\omega_1{}^2+4\alpha_1\alpha_2+\omega_2{}^2)^2-4(1-K^2)\omega_1{}^2\omega_2{}^2]}}{2(1-K^2)} \qquad (72)$$

and, on neglecting $\alpha_1\alpha_2$, we get

$$\beta^2 = \frac{(\omega_1{}^2 + \omega_2{}^2) \mp \sqrt{[(\omega_1{}^2 + \omega_2{}^2)^2 - 4(1-K^2)\omega_1{}^2\omega_2{}^2]}}{2(1-K^2)} \qquad (73)$$

From this equation we see that—

(i) The response angular velocity is not the same as for each circuit acting independently.

(ii) When $\omega_1 = \omega_2 = \Omega$, then the response angular velocity is $\beta = \Omega/\sqrt{(1 \pm K)}$. Thus the amount of difference depends on the value of K. For a graphical method illustrating the frequency variation due to change of K, reference should be made to Plummer.* When $K = 1$, $\beta_1 = \Omega/\sqrt{2}$, and $\beta_2 = 0$, when $K = 0$ then $\beta_1 = \beta_2 = \Omega$, i.e. the oscillations are of one frequency only.

(iii) The expression for the damping factor becomes

$$\gamma = (b\beta^2 + d)/(4a\beta^2 + 2c)$$
$$= \frac{(\alpha_1 + \alpha_2)\beta^2 + (\alpha_1\omega_2{}^2 + \alpha_2\omega_1{}^2)}{2(1 - K^2)\beta^2 + (\omega_1{}^2 + 4\alpha_1\alpha_2 + \omega_2{}^2)}$$

At resonance $\omega_1 = \omega_2 = \Omega$; then, on neglecting $\alpha_1\alpha_2$, we get

$$\gamma = \frac{\alpha_1 + \alpha_2}{2}\left[\frac{\beta^2 + \Omega^2}{(1 - K^2)\beta^2 + \Omega^2}\right]$$

With $\beta = \Omega/\sqrt{(1 \pm K)}$, then γ approximates to

$$\tfrac{1}{2}(\alpha_1 + \alpha_2)/(1 \pm K)$$

So, corresponding to the two values of β, there are two values of the damping coefficient γ.

We now have that

$$\left.\begin{aligned} i_1 &= A_1\varepsilon^{-\gamma_1 t}\cos(\beta_1 t + \lambda_1) + B_1\varepsilon^{-\gamma_2 t}\cos(\beta_2 t + \delta_1) \\ \text{and}\quad i_2 &= A_2\varepsilon^{-\gamma_1 t}\cos(\beta_1 t + \lambda_2) + B_2\varepsilon^{-\gamma_2 t}\cos(\beta_2 t + \delta_2) \end{aligned}\right\} \qquad (74)$$

where $A_1A_2B_1B_2$ are arbitrary constants. These may be shown not to be all independent.

* PLUMMER: *Phil Mag.* (1927), Vol. 34, p. 512.

On neglecting R_1, R_2 and substituting in equation (66), we get

$$\frac{A_1}{\beta_1}(\omega_1^2 - \beta_1^2)\sin(\beta_1 t + \lambda_1) + \frac{B_1}{\beta_2}(\omega_1^2 - \beta_2^2)\sin(\beta_2 t + \delta_1)$$
$$- \frac{M}{L_1\beta_1}A_2\beta_1^2\sin(\beta_1 t + \lambda_2) - \frac{M}{L_1\beta_2}B_2\beta_2^2\sin(\beta_2 t + \delta_2) = 0 \quad (75)$$

Equating terms of like frequency, we get

$$A_1(\omega_1^2 - \beta_1^2) = (M/L_1)A_2\beta_1^2 \text{ and } \lambda_1 = \lambda_2 = \lambda$$
$$B_1(\omega_1^2 - \beta_2^2) = (M/L_1)B_2\beta_2^2 \text{ and } \delta_1 = \delta_2 = \delta$$

So that we have

$$\left.\begin{aligned} i_1 &= A_1\cos(\beta_1 t + \lambda) + B_1\cos(\beta_2 t + \delta) \\ i_2 &= L_1/M\,\{[(\omega_1/\beta_1)^2 - 1]A_1\cos(\beta_1 t + \lambda) \\ &\qquad + \left[\left(\frac{\omega_1}{\beta_2}\right)^2 - 1\right]B_1\cos(\beta_2 t + \delta)\} \end{aligned}\right\} \quad (76)$$

When damping is considered, then the equations may be written as

$$\left.\begin{aligned} i_1 &= A_1\varepsilon^{-\gamma_1 t}\cos(\beta_1 t + \lambda) + B_1\varepsilon^{-\gamma_2 t}\cos(\beta_2 t + \delta) \\ \text{and } i_2 &= \frac{1}{K}\sqrt{(L_1/L_2)}\,\{[(\omega_1/\beta_1)^2 - 1]A_1\varepsilon^{-\gamma_1 t}\cos(\beta_1 t + \lambda) \\ &\qquad - [1 - (\omega_1/\beta_2)^2]B_1\varepsilon^{-\gamma_2 t}\cos(\beta_2 t + \delta) \end{aligned}\right\} \quad (77)$$

The general character of the oscillations may be deduced from equations (77). It is seen that there are two damped harmonic oscillations in each circuit, which leads to a periodic rise and fall of the amplitude corresponding to the acoustic beat between two musical notes.

Further, as $\beta_1 < \Omega$ while $\beta_2 > \Omega$, it is seen that the low frequency oscillations are in phase in the two circuits, while the higher frequencies are π radians out of phase. Thus the beats will be out of phase with the energy surging back and forth between the two circuits at a beat frequency of

$$(\beta_2 - \beta_1)/2\pi$$

If we assume at $t = 0$, $i_1 = i_2 = 0$, and that the charges on the condensers are $q_1 = q_0$ and $q_2 = 0$, it is easily shown that for the case of *tuned* circuits that

$$i_1 = \tfrac{1}{2}\frac{\Omega}{\sqrt{(1+K)}}q_0\varepsilon^{-\gamma_1 t}\sin\frac{\Omega}{\sqrt{(1+K)}}t \pm \tfrac{1}{2}\frac{\Omega}{\sqrt{(1-K)}}$$
$$q_0\varepsilon^{-\gamma_2 t}\sin\frac{\Omega}{\sqrt{(1-K)}}t$$

$$\text{and } i_2\sqrt{\frac{L_2}{L_1}} = \tfrac{1}{2}\frac{\Omega}{\sqrt{(1+K)}}q_0\varepsilon^{-\gamma_1 t}\sin\frac{\Omega}{\sqrt{(1+K)}}t - \tfrac{1}{2}\frac{\Omega}{\sqrt{(1-K)}} q_0\varepsilon^{-\gamma_2 t}\sin\frac{\Omega}{\sqrt{(1-K)}}t$$

So the effect of the coupling is to lower the frequency of the gravest tone and to raise it for the other.

5. Coupled Circuits in Forced Oscillation. When there is a voltage $E_0 \sin \omega t$ applied to the primary side, the circuit equations are

$$L_1pi_1 + R_1i_1 + (1/C_1p)i_1 + Mpi_2 = E\sin\omega t = E\varepsilon^{j\omega t}$$
$$L_2pi_2 + R_2i_2 + (1/C_2p)i_2 + Mpi_1 = 0$$

We will here investigate the steady state only, i.e. we assume that the circuit has been made long enough for steady-state conditions to obtain. The currents i_1 and i_2 will be sinusoidal, and so we get $i_1 = I_1\varepsilon^{j\omega t}$ and $i_2 = I_2\varepsilon^{j\omega t}$.

Hence $[R_1 + j(\omega L_1 - 1/\omega C_1)]I_1 + j\omega MI_2 = E$

and $[R_2 + j(\omega L_2 - 1/\omega C_2)]I_2 + j\omega MI_1 = 0$

or $Z_1I_1 + j\omega MI_2 = E$

and $Z_2I_2 + j\omega MI_1 = 0$

So that $I_1 = \dfrac{E}{Z_1 + \omega^2M^2/Z_2}$ and $I_2 = \dfrac{-j\omega ME}{Z_2[Z_1 + \omega^2M^2/Z_2]}$.

As the actual current is the imaginary part of I_1 and I_2, so setting

$$Z_1'\varepsilon^{j\phi} = Z_1 + \omega^2M^2/Z_2$$

Then $I_1 = (E/Z_1')\varepsilon^{-j\phi_1}$

or $i_1 = (E/Z_1')\sin(\omega t - \phi_1)$

and $I_2 = -(j\omega ME/Z_2'Z_1')\varepsilon^{-j(\phi_1+\phi_2)}$

or $i_2 = (\omega ME/Z_2'Z_1')\sin(\omega t - \phi_1 - \phi_2 - \pi/2)$

where $Z_1' = \sqrt{[(R_1 + \omega^2M^2R_2/Z_2^2)^2 + (X_1 - \omega^2M^2X_2/Z_2^2)^2]}$

$\tan\phi_1 = (X_1 - \omega^2M^2X_2/Z_2^2)/(R_1 + \omega^2M^2R_2/Z_2^2)$

$Z_2 = \sqrt{(R_2^2 + X_2^2)}$

$\tan\phi_2 = X_2/R_2$

6. Testing Transformers. Now let us consider the case of a transformer with a bushing of conductance g and capacity C

as burden. The circuit is as in Fig. 12. On neglecting the primary resistance and transferring all quantities to the secondary side, we get that*

$$-(M/L)e_1 = R_2 i_2 + \sigma L_2 p i_2 + e \quad . \quad . \quad (78)$$

But

$$i_2 = (g + Cp)e \quad . \quad . \quad . \quad (79)$$

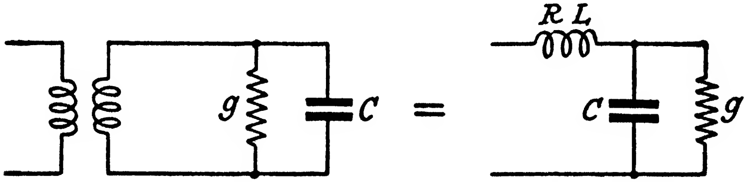

FIG. 12

and, on dropping subscripts, we get

$$E = (R + \sigma Lp)(g + Cp)e + e \quad . \quad . \quad (80)$$

where e is the voltage across the bushing and $\sigma = 1 - M^2/L_1L_2$.

Now E will be of the form $E \cos(\omega t - \theta)$, where θ is the "switching-in" angle, so that with $\sigma L = L$

$$e = \frac{E \cos(\omega t - \theta)}{LC[p^2 + (g/C + R/L)p + (1 + Rg)/(LC)]} \cdot (1) \quad . \quad (81)$$

Let the roots of the determinantal equation be given by $(-\alpha \pm \beta)$. Then, by the expansion theorem, we have, for the steady state, that

$$e_0 = \frac{E \cos(\omega t - \theta - \phi)}{\omega C\sqrt{\{R_2[1 + gL/CR]^2 + [(1 + Rg)/\omega C - \omega L]^2\}}} \quad . \quad (82)$$

where $\tan \phi = \dfrac{R[1 + gL/CR]}{(1 + Rg)/\omega C - \omega L}$.

The transient term will be given by

$$e_t = \varepsilon^{-j\theta} \sum_{n=1}^{n=\infty} \frac{E\varepsilon^{p_n t}}{(j\omega - p_n)Z'(p_n)} \quad . \quad . \quad (83)$$

where $Z'(p) = [2p \pm (g/C + R/L)]LC = \pm 2LC\beta$.

Let $\alpha = g/2C + R/2L$

and $\beta = \sqrt{[(g/2C + R/2L)^2 - (1 + Rg)/LC]}$

* RUSSELL: *A.C. Theory*, Vol. II, p. 260. C.U.P. 1914.

Then we get

$$e_t = \frac{E\varepsilon^{-\alpha t}\varepsilon^{-j\theta}}{2LC\beta}\left[\frac{\varepsilon^{\beta t}}{j\omega+\alpha-\beta} - \frac{\varepsilon^{-\beta t}}{j\omega+\alpha+\beta}\right]$$

$$= \frac{E\varepsilon^{-\alpha t}\varepsilon^{-j\theta}}{2LC\beta}\left[\frac{(j\omega+\alpha)\sinh\beta t + \beta\cosh\beta t}{(\alpha^2-\beta^2-\omega^2)+j(2\alpha\omega)}\right] \quad . \quad (84)$$

On rationalizing and discarding imaginaries, we get

$$e_t = \frac{E\varepsilon^{-\alpha t}}{LC\beta}$$

$$\left[\frac{\cos(\theta+\psi).(\alpha\sinh\beta t+\beta\cosh\beta t)+\omega\sin(\theta+\psi).\sinh\beta t}{\sqrt{[(\alpha^2-\beta^2-\omega^2)^2+4\alpha^2\omega^2]}}\right] (85)$$

$$\text{where } \tan\psi = \frac{2\omega\alpha}{\alpha^2-\beta^2-\omega^2} = \frac{L^2}{R}\left[\frac{1+gL/CR}{(1+Rg)/C\omega-\omega L}\right]$$

$$= \frac{L^2}{R^2}\tan\phi$$

When the resistance R is small, then the value of β will approximate to $j/\sqrt{(LC)}$, so that $\tan\phi = 0$ or $\phi = 0$. With a high natural frequency we may neglect terms containing ω/β and α/β, so that we get

$$e = E_0[\cos(\omega t - \theta) - \varepsilon^{-\alpha t}\cos\theta\,.\,\cos\beta t] \quad . \quad (86)$$

The maximum voltage will occur when the switch is closed at $\omega t = \theta = 0$, i.e. at maximum applied volts. Further, due to the value of circuit constants, a voltage higher than requisite may be applied unwittingly to the bushing.

7. **Arc Shunted by Resistance.** As the current in an a.c. arc must follow the voltage fluctuations, the arc will be extinguished and restruck periodically. The phenomenon will depend on the rate of building up of the voltage across the arc space. We consider a circuit as shown in Fig. 13. Let the current through the inductance be i_L, the current through the capacity be i_c, and the total current be I. For a voltage applied to the circuit we get, when the arc current is zero—

$$\left.\begin{aligned} E &= RI + (1/Cp)i_c = RI + q_c/C \\ I &= i_L + i_c = i_L + pq_c \end{aligned}\right\} \quad . \quad . \quad (87)$$

$$\therefore pI = pi_L + p^2q_c$$

Combining these equations, we get that

$$E = (q_c/C) + R[i_L + pq_c]$$

On differentiation this becomes

$$0 = p(q_c/C) + Rpi_L + Rp^2q_c$$

But
$$Lpi_L = q_c/C$$

$$\therefore Rp^2q_c + (1/C)pq_c + (Rq_c/LC) = 0$$

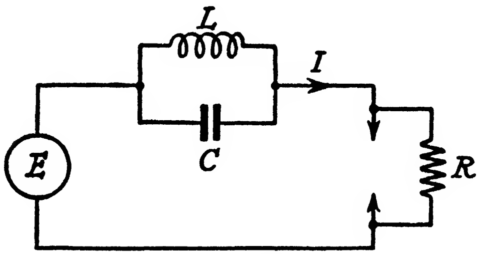

FIG. 13

The roots of this equation are given by

$$p = -1/2CR \pm \sqrt{(1/4C^2R^2 - 1/LC)}$$

so that if $1/4C^2R^2 < 1/LC$, or R is greater than the critical resistance, we get as solution—

$$q_c = \varepsilon^{-t/2CR}(A \cos \omega t + B \sin \omega t) \quad . \quad . \quad (88)$$

where $\omega = \sqrt{(1/LC - 1/4C^2R^2)}$.

Thus the voltage across the arc is

$$e_a = E - q_c/C = E - \varepsilon^{-t/2CR}[A \cos \omega t + B \sin \omega t]/C \quad (89)$$

The constants may be determined from terminal conditions at $t = 0$, as, for example, $e_a = E_1$ and $i_c = -I_1$, where I is the current at which the arc fails. So that $E_1 = E - A/C$.

The condenser current is $i_c = pq_c$, so we have

$$-I_1 = B\omega - A/2CR$$

$$\therefore A = C[E - E_1] \text{ and } B = [-I_1 + (E - E_1)/2R]/\omega$$

and
$$e_a = E - \varepsilon^{-t/2CR}[(E - E_1) \cos \omega t + (1/\omega C)$$
$$[(E - E_1)/2R - I_1] \sin \omega t \quad . \quad . \quad . \quad (90)$$

From similar considerations it may be shown that if $1/LC < 1/4R^2C^2$, i.e. if the resistance is less than the critical value, the voltage across the arc is

$$e_a = E - \varepsilon^{-t/2CR}\left[(E - E_1) \cosh \beta t + (1/C\beta)\left(\frac{E - E_1}{2R} - I_1\right) \sinh \beta t\right] \quad . \quad . \quad . \quad (91)$$

On differentiating these expressions, the rate of building up of the voltage across the arc space may be determined. We have

$$de_a/dt = [I_1/C]_{t=o} \quad . \quad . \quad . \quad (92)$$

Thus it is seen that the initial rate of voltage recovery is inversely proportional to the capacity of the system and directly proportional to the current at which the arc fails.

CHAPTER IV*

BOUNDARY CONDITIONS

1. **Mathematical Theorem: Subsidiary Equation.** So far we have considered cases where the circuit was dead at a time $t = 0$. As this condition does not always hold, we will now give a subsidiary equation which in combination with the standard form of the expansion theorem gives the solution under these conditions.

Let the general equation be given by

$$F(t) = \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + a_2 \frac{d^{n-2}y}{dt^{n-2}} + \ldots + a_n y$$

$$= p^n y + a_1 p^{n-1} y + a_2 p^{n-2} y + \ldots + a_n y \quad . \quad (93)$$

At $t = 0$ let $p^{n-1}y = c_1$, $p^{n-2}y = c_2$, and so on. Also let $p^n y = u(t)$.

Then by integration we obtain

$$p^{n-1}y = \int u(t)dt + c_1$$

Repeating this process, we get

$$p^{n-2}y = \int\int u(t)d^2t + c_1 t + c_2$$
$$= p^{-2}u(t) + c_1 t + c_2$$

So $y = p^{-n}u(t) + c_1 t^{n-1}/(n-1)! + c_2 t^{n-2}/(n-2)!$
$+ \ldots + c_n \quad . \quad . \quad . \quad . \quad . \quad (94)$

Substituting in equation (93), we have

$$F(t) = u(t) + a_1 p^{-1}u(t) + a_2 p^{-2}u(t) \ldots$$
$$+ a_n p^{-n}u(t) + c_1 a_1 + a_2(c_1 t + c_2) \ldots$$
$$= u(t)[1 + a_1 p^{-1} + a_2 p^{-2} \ldots + a_n p^{-n}] + c_1 a_1$$
$$+ a_2(c_1 t + c_2) \ldots + a_n(c_1 t^{n-1}/(n-1)!$$
$$+ c_2 t^{n-2}/(n-2)! \ldots) \quad . \quad . \quad . \quad . \quad (95)$$

By equation (94), we have

$$u(t) = p^n[y - c_1 t^{n-1}/(n-1)! - c_2 t^{n-2}/(n-2)! \ldots - c_n]$$

So by equation (95) we get

$$F(t) = [p^n + a_1 p^{n-1} + \ldots + a_n]\,[y - c_1 t^{n-1}/(n-1)! - \ldots$$
$$- c_n] + a_1 c_1 + \ldots + a_n[c_1 t^{n-1}/(n-1)! + \ldots] \quad (96)$$

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Let $\qquad f(c) = c_1 t^{n-1}/(n-1)! + c_2 t^{n-2}/(n-2)! + \ldots$

Then $\qquad p^n f(c) = c_1 p + c_2 p^2 \ldots + c_n p^n$

and $\qquad a_1 p^{n-1} f(c) = (a_1 c_1 + a_1 c_2 p + \ldots)$

$$\ldots \qquad \ldots$$

Thus equation (96) reduces to

$$F(t) = [p^n + a_1 p^{n-1} + a_2 p^{n-2} \ldots + a_n]y$$
$$- c_1 p - c_2(p^2 + a_1 p) + \ldots$$

$$\therefore y = \frac{F(t) + c_1 p + c_2(p^2 + a_1 p) + \ldots}{p^n + a_1 p^{n-1} + a_2 p^{n-2} \ldots + a_n} \quad . \quad . \quad (97)$$

We will demonstrate the use of this equation by a few examples.

2. **Breaking Circuit.** Let us consider an RL circuit. Initially, the current will be I at $t = 0$, and the voltage equation at break is given by $Ri + Lpi = 0$ or $pi + (R/L)i = 0$. From equation (97) we have that $F(t) = 0$ and $c_1 = I$. So that we get

$$i = I \frac{p}{p + R/L} \cdot (1)$$

On expansion this becomes

$$i = I\varepsilon^{-(R/L)t}$$

For the RLC circuit we have the following voltage equation—

$$Ri + Lpi + (1/Cp)i = 0$$

or

$$p^2 q + (R/L)pq + (1/LC)q = 0$$

At $t = 0$, $q = Q$, and we have that $F(t) = 0$, $a_1 = R/L$, $a_2 = 1/LC$, $c_1 = 0$, and $c_2 = Q$ as the constants of equation (97). Thus equation (97) now becomes

$$q = \frac{Q(p^2 + Rp/L)}{p^2 + Rp/L + 1/LC} (1)$$

By the expansion theorem this becomes

$$q = Q\left[\frac{(p_1 + R/L)\varepsilon^{p_1 t}}{2p_1 + R/L} + \frac{p_2 + R/L)\varepsilon^{p_2 t}}{2p_2 + R/L}\right]$$

The values of the roots are

$$p = -R/2L \pm \sqrt{(R^2/4L^2 - 1/LC)} = -\alpha \pm \omega_0, \text{ say.}$$

Hence $\qquad q = Q\varepsilon^{-\alpha t}[\alpha \sinh \omega_0 t + \omega_0 \cosh \omega_0 t]/\omega_0$

For oscillating conditions we have $\omega_0 = j\omega$, say, and

$$q = [Q \,.\, \varepsilon^{-\alpha t} \,.\, \cos(\omega t - \phi)]/\omega\sqrt{(\alpha^2 + \omega^2)}$$

where $\tan\phi = \alpha/\omega$.

And $$pq = i = -(Q\varepsilon^{\alpha t}\sin\omega t)/\omega_0(\alpha^2 + \omega_0{}^2)$$

3. **Field Discharge Resistance.** When the field circuit of a machine is broken, the energy stored in the magnetic field must be dissipated so that the sparking at the switch contacts is minimized. Usually this is done by opening the circuit through a discharge resistance R_2 (Fig. 14) which is inserted in the circuit before the main circuit is broken. So we have that

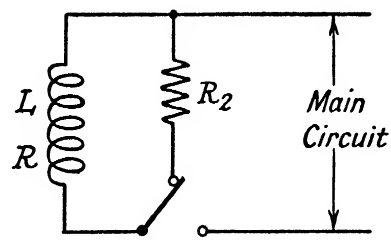

FIG. 14

$$(R + R_2)i + Lpi = 0$$

i.e. $$pi + (R + R_2)i/L = 0$$

At $t = 0$ let $i = I$. So, neglecting the presence of iron, we get

$$i = I\frac{p}{p + (R + R_2)/L}(1) \doteqdot I\varepsilon^{-(R+R_2)t/L} \qquad (98)$$

The energy stored in the magnetic field is given by $\frac{1}{2}LI^2$ and the energy dissipated in the circuit by resistance is

$$(R + R_2)I^2\varepsilon^{-2\alpha t}$$

So that the net energy in the spark will be

$$I^2[\tfrac{1}{2}L - (R_1 + R_2)\varepsilon^{-2\alpha t}]$$

For no sparking we must have a time of switch opening

$$t = \frac{1}{2\alpha}\,\mathrm{lgn.}\frac{L}{2(R + R_2)}\ \mathrm{sec.} \qquad . \qquad . \quad (99)$$

4. **Automatic Traffic Signals.** A further example of the discharge circuit is in traffic-signal systems. An elementary diagram is shown in Fig. 15, where A is the relay operated by vehicles having the right of way and B is a demand relay for vehicles wanting the road. With the relay A on the upper contact the condenser is charging, so that

$$i = \frac{E}{R_1 + pC}(1) \doteqdot \frac{E}{R_1}\varepsilon^{-t/CR_1} \qquad . \qquad . \ (100)$$

and the voltage across the condenser is

$$\int_0^t (i/C)dt = E[1 - \varepsilon^{-t/CR_1}] \qquad . \qquad . (101)$$

When the relay B closes, the voltage across the neon tube is

$$E - E[1 - \varepsilon^{-t/CR_1}] = E\varepsilon^{-t/CR_1} \qquad . \qquad . (102)$$

which is insufficient to cause the neon tube to fire. After a definite time the relay A is moved to the lower contact by a master timer and the condenser discharges through R_2, giving

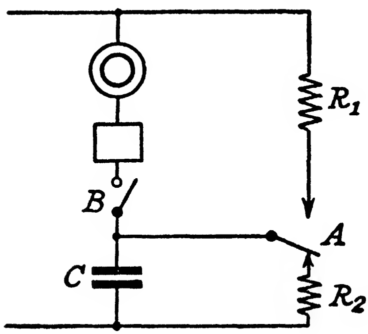

FIG. 15

$$i_2 = (E/R_2)\varepsilon^{-t/CR_2} \qquad . (103)$$

The voltage across the condenser is $(1/Cp)i_2 = E\varepsilon^{-t/CR_2}$ and across the neon tube is $E[1 - \varepsilon^{-t/CR_2}]$. When this voltage is equal to the critical volts e_0, the tube fires and the time of discharge is

$$t = CR_2 \operatorname{lgn.} [E/(E - e_0)]$$

The discharge of the neon tube actuates a relay, which in turn operates the signal light. The size of the condenser is the limiting factor in the time equation, and should be such that it will not allow the neon tube to fire before the last vehicle is clear of the intersection.*

5. Opening of a Circuit through a Condenser. Frequently condensers are placed across the contacts of a switch to reduce sparking. In Fig. 16 is shown an LR circuit with a condenser across the switch contacts. Initially, the current with the switch closed is steady and equal to $i_0 = E/R$.

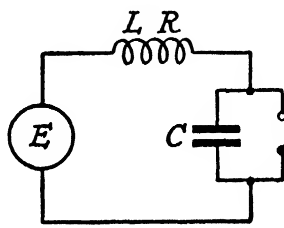

FIG. 16

On opening the switch the condenser is inserted. So, neglecting arc phenomena at switch contacts, we get

$$i = \frac{ECp}{LC[p^2 + Rp/L + 1/LC]} \cdot (1)$$

At $t = 0$, $i = i_0$ *or* $q = 0$ and $pq = i_0$. Thus we get $c_1 = i_0$ and so

$$q = \frac{E/L + pi_0}{p^2 + Rp/L + 1/LC} \cdot (1)$$

* PRIEST: *Journ. I.E.E.* (1935), Vol. 77, p. 149.

$$\therefore\ i_c = \frac{Ep/L + p^2 i_0}{p^2 + Rp/L + 1/LC} \cdot (1) \qquad . \quad . \quad . \ (104)$$

$$\doteqdot \frac{E}{L}\varepsilon^{-\alpha t}\frac{\sin \omega t}{\omega} - \frac{E}{R}\frac{\omega_0}{\omega}\varepsilon^{-\alpha t}\sin(\omega t - \phi) \qquad . \ (105)$$

where $\tan\phi = \omega/\alpha$.

At $t = 0$ the current through the condenser $= E/R$.

Initially all the current passes through the condenser and none through the switch contacts, and thus the voltage drop across the contacts is small.

6. **Restriking Voltage subsequent to Circuit Interruption.** When a circuit-breaker opens, an arc is established between the contacts. On an a.c. circuit, every time the current wave passes through zero the arc is extinguished for a period of 100 or so microseconds. Should the dielectric strength of the gap be re-established at sufficient speed, then the voltage across the gap will be insufficient to cause breakdown. The arc would then be extinguished. Consideration should also be given to the effect of power factor. For example, on short circuit the current lags by 90° approximately on the voltage. When the current is zero, the applied voltage will be at its maximum value. Thus we have the maximum generator volts maintaining the arc. The voltage at the moment the switch breaks may be termed the *recovery* volts.

Oscillograms of the current under short-circuit conditions show high-frequency oscillations. It would appear that the ability of a breaker to interrupt a circuit depends on the increase in the dielectric strength of the gap occurring at a greater rate than the rise of these high-frequency oscillations. This voltage is termed the *restriking voltage*, and may be calculated as follows.

The opening of a switch is equivalent to the introduction into circuit of an e.m.f. equal in magnitude but opposite in sign to the potential difference appearing at the switch. Let $Z(p)$ be the impedance operator of the system and I the current prior to the opening of the switch. Then the voltage across the switch contacts would be $I \,.\, Z(p) = e$. This is the voltage necessary to prevent current flow. To calculate $Z(p)$ we have to take into account the circuit constants, including the arc. For a power arc the drop is directly proportional to *current* × *arc length.* Assuming that the switch plunger bar reaches its final position instantaneously, then the arc drop is directly

proportional to the current. So for one phase the circuit may be represented as Fig. 17, where RLC are the circuit constants (including those of the machine) and r the arc resistance. We get

$$Z(p) = \frac{1}{1/r + Cp + 1/(R + Lp)} \cdot (1)$$

$$\therefore\ e = I \,.\, Z(p) = I \,.\, \frac{r(R + Lp)}{LCrp^2 + (RCr + L)p + R + r} \cdot (1) \quad (106)$$

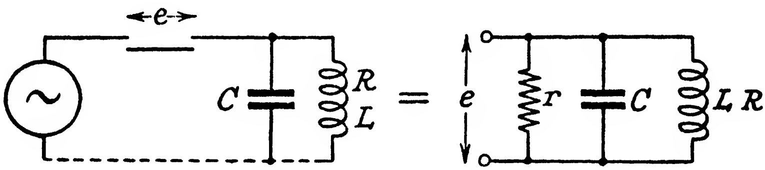

FIG. 17

As e is the voltage across the switch contacts it may be termed the *restriking voltage*. If $\alpha = (RCr + L)/2LCr$ and $\omega_0{}^2 = (R + r)/LCr$, then we have

$$p = -\alpha \pm \sqrt{(\alpha^2 - \omega_0{}^2)} = -\alpha \pm \omega_1, \text{ say.}$$

From the expansion theorem we get

$$e = I\left[\frac{rR}{R + r} + \frac{\varepsilon^{-\alpha t}}{\omega_1 C}\sinh \omega_1 t + \frac{R\varepsilon^{-\alpha t}}{LCr\omega_1}\left(\frac{\omega_1 \cosh \omega_1 t + \alpha \sinh \omega_1 t}{\alpha^2 + \omega_1{}^2}\right)\right] \quad (107)$$

For an a.c. system we have that

$$e = I\left[\frac{r(Lp + R)}{LCrp^2 + (RCr + L)p + R + r} \cdot \cos \omega t\right] \cdot (1)$$

The steady-state term is

$$e_s = \frac{Ir}{2}\left[\frac{(R + j\omega L)\varepsilon^{j\omega t}}{-\omega^2 LCr + (RCr + L)j\omega + R + r}\right] \quad (108)$$

and the transient term is

$$e_t = Ir \sum_n \frac{p_n(R + Lp_n)\varepsilon^{p_n t}}{(p_n{}^2 + \omega^2)(2LCrp_n + RCr + L)}$$

$$= Ir \sum_n \frac{p_n(R + Lp_n)\varepsilon^{p_n t}}{2LCr\omega_1(p_n{}^2 + \omega^2)} \quad (109)$$

Now, p_n is usually very large compared with ω, and so we may approximate—

$$e_t = Ir \sum_n \frac{(R + Lp_n)\varepsilon^{p_n t}}{2LCr\omega_1 p_n} \qquad . \quad . \quad . \quad . \quad . \quad (110)$$

On neglecting circuit resistance and damping, we get

$$e_t = \frac{I\varepsilon^{-\alpha t}}{2LC\omega_1} \cdot L(\varepsilon^{\omega_1 t} - \varepsilon^{-\omega_1 t}) = \frac{I\varepsilon^{-\alpha t}}{C\omega_1} \sinh \omega_1 t \qquad . \quad (111)$$

Had we considered the circuit as having distributed constants, then, as will be seen in Chapter VII, there are several free periods, and the expression is in the form

$$e = e_s + \sum_{n=1}^{n=\infty} \frac{I_n \varepsilon^{-\alpha t}}{C\omega_n} \sinh \omega_n t \qquad . \quad . \quad . \quad . \quad (112)$$

Further investigation on these lines may prove of some value in the rating of circuit-breakers for a definite system. The difficulties respecting the calculation of the natural periods of a system are not insuperable.

REFERENCES

PARK AND SKEATS: "Circuit-breaker Recovery Voltages," *Trans. A.I.E.E.* (1931), Vol. 50, gives excellent oscillograms and develops ideas more completely.

"Switching Surges with Transformer Load Ratio Contactors," *Trans. A.I.E.E.* (1937), Vol. 56, p. 1464.

CHAPTER V

SINUSOIDAL AND DAMPED VOLTAGES
OTHER METHODS

We have already demonstrated the expansion method of dealing with sinusoidal voltages (Chapter II). When we come to deal with more complicated voltage expressions, that method becomes unwieldy; but we are able to make some simplifications. These are—

1. To express the voltage function as its equivalent operator.
2. To use the Heaviside shifting method.
3. To use the superposition theorem or the Duhamel integral.

1. **Equivalent Operator of the Voltage Function.** This method is based on the following mathematical argument—

$$\cos \omega t = \tfrac{1}{2}(\varepsilon^{j\omega t} + \varepsilon^{-j\omega t})$$

But $\varepsilon^{j\omega t}$ expressed operationally is $\dfrac{p}{p - j\omega} \cdot (1)$ (see Appendix), and $\varepsilon^{-j\omega t}$ is $\dfrac{p}{p + j\omega} \cdot (1)$.

Hence

$$\cos \omega t \doteqdot \tfrac{1}{2}\left[\frac{p}{p + j\omega} + \frac{p}{p - j\omega}\right] \cdot (1) \doteqdot \frac{p^2}{p^2 + \omega^2} \cdot (1). \quad (113)$$

In like manner we obtain

$$\sin \omega t \doteqdot \frac{\omega p}{p^2 + \omega^2} \cdot (1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

Also $\cos(\omega t + \beta) = \cos \omega t \cos \beta - \sin \omega t \sin \beta$.

Hence, on substituting equivalencies, we get

$$\cos(\omega t + \beta) \doteqdot \frac{p^2 \cos \beta - \omega p \sin \beta}{p^2 + \omega^2} \cdot (1)$$

For the hyperbolic functions we have

$$\left.\begin{aligned} \sinh \omega t &\doteqdot \frac{\omega p}{p^2 - \omega^2} \cdot (1) \\ \text{and } \cosh \omega t &\doteqdot \frac{p^2}{p^2 - \omega^2} \cdot (1) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

2. The Shifting Method. This method was introduced by Heaviside,* and he demonstrates it as follows—

$$p[u\varepsilon^{-\alpha t}] = \varepsilon^{-\alpha t}\,(du/dt) - \varepsilon^{-\alpha t}\,\alpha u = \varepsilon^{-\alpha t}(p-\alpha)u \quad . \quad (116)$$

It is thus seen that the rule for changing the sequence of p and $\varepsilon^{-\alpha t}$ is that on moving p from the left to the right of $\varepsilon^{-\alpha t}$ we must change the p to $(p-\alpha)$; on moving to the left we change p to $p+\alpha$. Actually we are multiplying $f(p)$ by either $\varepsilon^{-\alpha p}$ or $\varepsilon^{+\alpha p}$, and so delaying or advancing the time of application of the voltage as shown in Fig. 45. Reversing the equation, this theorem may be written as

$$\varepsilon^{-\alpha t}\cdot\frac{1}{Z(p)}\cdot(1) = \frac{1}{Z(p+\alpha)}\cdot\varepsilon^{-\alpha t}\cdot(1)$$

$$\risingdotseq \frac{1}{Z(p+\alpha)}\cdot\frac{p}{p+\alpha}\cdot(1) \quad . \quad . \quad (117)$$

Thus, for the expression $\varepsilon^{-\beta t}\sin\omega t\,.\,(1)$, we have that

$$\sin\omega t \risingdotseq \frac{\omega p}{p^2+\omega^2}$$

so that

$$\varepsilon^{-\beta t}\sin\omega t \risingdotseq \varepsilon^{-\beta t}\cdot\frac{\omega p}{p^2+\omega^2}\cdot(1) \risingdotseq \frac{\omega(p+\beta)}{(p+\beta)^2+\omega^2}\cdot\varepsilon^{-\beta t}\cdot(1)$$

$$\text{i.e. } \varepsilon^{-\beta t}\sin\omega t \risingdotseq \frac{\omega(p+\beta)}{(p+\beta)^2+\omega^2}\cdot\frac{p}{p+\beta}\cdot(1) \quad . \quad . \quad (118)$$

$$\risingdotseq \frac{\omega p}{(p+\beta)^2+\omega^2}\cdot(1) \quad . \quad . \quad . \quad . \quad (119)$$

Similarly

$$\varepsilon^{-\beta t}\cos\omega t \risingdotseq \frac{p(p+\beta)}{(p+\beta)^2+\omega^2}\cdot(1) \quad . \quad . \quad . \quad . \quad (120)$$

A further example, taken from line theory, will suffice to show the method. The operator is

$$\varepsilon^{-x/v}\sqrt{[(p+\rho)^2-\sigma^2]}\,.\,(1) \quad . \quad . \quad (121)$$

Multiplication by $\varepsilon^{-\rho t}\,.\,\varepsilon^{\rho t}$ does not alter the value of the operator, so we get

$$\varepsilon^{-\rho t}\,\varepsilon^{\rho t}\,\varepsilon^{-x/v}\sqrt{[(p+\rho)^2-\sigma^2]}$$

$$= \varepsilon^{-\rho t}\,\varepsilon^{-x/v}\sqrt{(p^2-\sigma^2)}\,\varepsilon^{\rho t}\,.\,(1) \quad . \quad (122)$$

* O. Heaviside: *Electromagnetic Theory*, Vol. II, p. 293.

3. **Superposition Theorem or Duhamel Integral.** This method is used by Carslaw in his *Theory of Fourier's Series.* Fundamentally it is a "step by step" method, and may be derived in the following manner—

At a time t let an e.m.f. $E(t)$ be applied to a circuit whose admittance is $A(t)$. Then the current $i(t) = E(t) . A(t)$, where $A(t)$ is $Y(p)/Z(p)$.

If now a voltage $E(\lambda)$ is applied to the circuit at a time $t = \lambda$, we then have

$$i(t - \lambda) = E(\lambda) . A(t - \lambda) \quad . \quad . \quad (123)$$

as the additional current. So we would get a total response (Fig. 18) of

$$E(0) . A(t) + E(\lambda) . A(t - \lambda) + E(2\lambda) . A(t - 2\lambda) + . . . \quad (124)$$

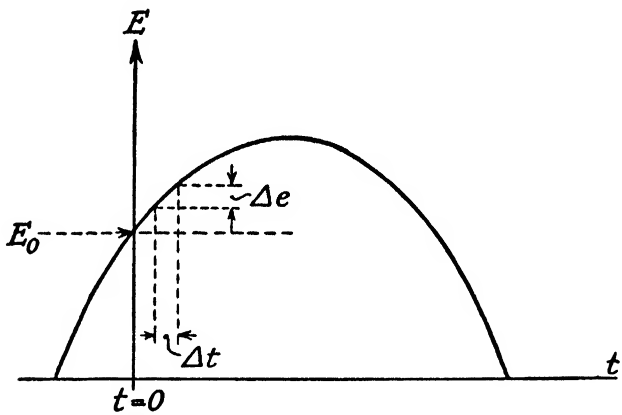

FIG. 18

If $E(\lambda) = \Delta E$ is applied at a time Δt, then the above equation is

$$E(0) . A(t) + \Delta E . A(t - \Delta t) + \Delta E\, A(t - 2\Delta t) + . . . \quad (125)$$

or, in the limit, we get

$$I(t) = E(0) . A(t) + \int_0^t A(t - dt) . (dE/dt)\, dt$$

i.e. $$I(t) = d/dt \int_0^t E(u) . A(t - u)\, du \quad . \quad . \quad . \quad (126\text{A})$$

or $$= d/dt \int_0^t E(t - u) . A(u)\, du \quad . \quad . \quad . \quad (126\text{B})$$

By using the well-known partial integration formula

$$\Big[XY\Big]_0^t = \int_0^t XdY + \int_0^t YdX$$

and noting that, if $\phi(t) = \int_{t_0}^{t_1} f(t,\lambda)\, d\lambda$, then

$$\frac{d\phi}{dt} = \int_{t_0}^{t_1} \frac{df}{dt} d\lambda - f(t,t_0)\frac{dt_0}{dt} + f(t,t_1)\frac{dt_1}{dt}$$

we may transform our equation into one of the following forms, the exact form to be used in any particular analysis depending on the problem and the ease with which the mathematics may be handled.

$$I(t) = E(0) \,.\, A(t) + \int_0^t A(t-u) \,.\, E'(u)\, du \quad . \quad (126\text{C})$$

$$= A(0) \,.\, E(t) + \int_0^t E(t-u) \,.\, A'(u)\, du \quad . \quad (126\text{D})$$

$$= E(t) \,.\, A(0) + \int_0^t A'(t-u) \,.\, E(u)\, du \quad . \quad (126\text{E})$$

$$= E(0) \,.\, A(t) + \int_0^t E'(t-u) \,.\, A(u)\, du \quad . \quad (126\text{F})$$

4. **Applications of the Above Methods.** We will now show some applications of the above methods. As all equations must lead to the same result, we will give only one example of each method.

METHOD 1. Let us consider the sudden short circuit of an alternator. In an elementary way we may represent the machine by resistance and inductance of constant values and neglect all other effects. When the machine is suddenly short-circuited by an ideal switch, we have

$$Ri + Lpi = E \sin(\omega t + \alpha) \,.\, (1)$$

or

$$i = \frac{E}{R + Lp} \sin(\omega t + \alpha) \,.\, (1) \quad . \quad . \quad (127)$$

Now, using equation (114), we get

$$i = \frac{E}{R + Lp} \cdot \frac{\omega p \cos\alpha + p^2 \sin\alpha}{p^2 + \omega^2} \,.\, (1)$$

The impedance operator is $(R + Lp)(p^2 + \omega^2)$, and it will have roots $- R/L$ and $\pm j\omega$ respectively.

On applying the expansion theorem (Form 1) we note that

$$dZ(p) = Z_1(p) \, . \, Z_2'(p) + Z_1'(p) \, . \, Z_2(p) \quad . \quad (128)$$

where $Z_1 = R + Lp$ and $Z_2(p) = p^2 + \omega^2$.

On substitution of the roots obtained from $Z_1(p) = 0$, it is seen that the first term will cancel out, leaving the second term $Z_2(p) \, . \, Z_1'(p)$. Similarly, on the substitution of those obtained from $Z_2(p) = 0$, the first term only will remain. (This is an alternative to Bórel's method, Chapter I.) Thus we get

$$i = E\left[\frac{j\omega^2 \cos \alpha - \omega^2 \sin \alpha}{-2\omega^2(R + j\omega L)} \varepsilon^{+j\omega t} + \frac{-j\omega^2 \cos \alpha - \omega^2 \sin \alpha}{-2\omega^2(R - j\omega L)} \varepsilon^{-j\omega t} \right.$$
$$\left. + \frac{\omega \cos \alpha - (R/L) \sin \alpha}{L[(R/L)^2 + \omega^2]} \varepsilon^{-(R/L)t} \right] . \quad . \quad (129)$$

Let $\tan \phi = \omega L/R$ and $Z^2 = R^2 + \omega^2 L^2$. Then we have

$$i = \frac{E}{Z}\left[\frac{\cos (\alpha - \phi) + j \sin (\alpha - \phi)}{2j} \varepsilon^{j\omega t} \right.$$
$$\left. - \frac{\cos (\alpha - \phi) - j \sin (\alpha - \phi)}{2j} \varepsilon^{-j\omega t} - \sin (\alpha - \phi) \varepsilon^{-(R/L)t} \right]$$
$$= (E/Z)[\cos (\alpha - \phi) \sin \omega t - \sin (\alpha - \phi) \cos \omega t$$
$$- \sin (\alpha - \phi) \, . \, \varepsilon^{-(R/L)t}]$$
$$= (E/Z) [\sin (\omega t + \phi - \alpha) - \sin (\alpha - \phi) \, . \, \varepsilon^{-(R/L)t}] \quad . \quad (130)$$

In this case ωL will be very large compared with R, so that ϕ approaches 90°. The equation then reduces to

$$i = (E/\omega L) [\cos (\omega t - \alpha) - \varepsilon^{-(R/L)t} . \cos \alpha] \quad . \quad . \quad (131)$$

If $\phi = \alpha$, then we have $i = (E/\omega L) \sin \omega t$, i.e. there is no current transient term when the switch is closed at $\alpha \cong \pi/2$ or when the supply voltage is a maximum.

For the maximum transient the switch must be closed when $\cos \alpha$ is a maximum, i.e. when $\alpha = 0$, or the supply voltage is zero. The subject will be dealt with in more detail in Chapter VI.

Methods 2 and 3. For the case when the applied voltage is

of the form $E\varepsilon^{-\lambda t}\sin \omega t$, we apply the shifting method first and then use the Duhamel integral. Thus we have

$$i = \frac{E}{R + Lp} \cdot \varepsilon^{-\lambda t} \sin \omega t \,.\, (1)$$

$$= \varepsilon^{-\lambda t} \frac{E}{R + L(p - \lambda)} \,.\, \sin \omega t \,.\, (1) \quad . \quad (132)$$

Now, applying the Duhamel integral, we get

$$A(t) = [1 - \varepsilon^{(\lambda - \alpha)t}]/(R - \lambda L)$$

where

$$\alpha = R/L$$
$$A(0) = 0$$
$$E(u) = E \sin \omega u$$

So that we have

$$\varepsilon^{\lambda t} \,.\, i(t) = A(0) \,.\, E(t) + \int_0^t E(t - u) \,.\, A'(u)\, du$$

$$= \frac{E}{L} \int_0^t \varepsilon^{(\lambda - \alpha)u} \sin \omega (t - u)\, du \quad . \quad . \quad . \quad (133)$$

The value of the integral is

$$\varepsilon^{(\lambda - \alpha)t}\left[\frac{\omega}{\omega^2 + (\lambda - \alpha)^2}\right] - \frac{1}{\omega^2 + (\lambda - \alpha)^2}$$
$$[(\lambda - \alpha) \sin \omega t + \omega \cos \omega t] \quad . \quad (134)$$

So that

$$i = \frac{E}{L}\left[\frac{\omega \varepsilon^{-\alpha t}}{\omega^2 + (\lambda - \alpha)^2} - \frac{\varepsilon^{-\lambda t}}{\sqrt{[\omega^2 + (\lambda - \alpha)^2]}} \sin (\omega t + \phi)\right] \quad . \quad . \quad (135)$$

where $\tan \phi = \omega/(\lambda - \alpha) = \omega L/(\lambda L - R)$

$$\therefore\ i = \frac{E}{\sqrt{[\omega^2 L^2 + (\lambda L - R)^2]}}$$
$$[\varepsilon^{-\alpha t} \sin \phi - \varepsilon^{-\lambda t} \sin (\omega t + \phi)] \quad . \quad (136)$$

METHOD 3. For the application of the third method we choose a circuit (RL) to which a damped wave is applied of the form $E\varepsilon^{-\lambda t}$.

Now, for a wave given by $E\varepsilon^{-\lambda t}$ we get

$$i = \frac{E}{R + Lp} \cdot \varepsilon^{-\lambda t}\, (1) \quad . \quad . \quad . \quad (137)$$

For a constant e.m.f. the equivalent of $1/(R + Lp)$ is $[1 - \varepsilon^{-\alpha t}]/R$, so that

$$A(t) = [1 - \varepsilon^{-\alpha t}]/R$$

and
$$E(t) = E\varepsilon^{-\lambda t}$$

Thus we have $A(0) = 0$ and $A'(u) = (\alpha/R)\varepsilon^{-\alpha u}$.

Then $I(t) = A(0)E(t) + \int_0^t E(t-u) \,.\, A'(u)\,du$

or
$$I(t) = \int_0^t (E\alpha/R)\varepsilon^{-\alpha u} \,.\, \varepsilon^{-\lambda(t-u)}\,du \quad . \quad . \quad . \quad (138)$$

and
$$I(t) = (E\alpha/R)\varepsilon^{-\lambda t}\int_0^t \varepsilon^{-u(\alpha-\lambda)}\,du$$
$$= (E\alpha/R) \,.\, \varepsilon^{-\lambda t}[\varepsilon^{-t(\alpha-\lambda)} - 1]/(\lambda - \alpha)$$

i.e.
$$I(t) = [E/(\lambda L - R)\,[\varepsilon^{-\alpha t} - \varepsilon^{-\lambda t}] \quad . \quad . \quad . \quad (139)$$

5. **The *RLC* Circuit with Applied Damped Sine-wave Voltage.** In radio work, oscillatory e.m.f.s are impressed on *RLC* circuits. If the applied voltage be $E\varepsilon^{-\lambda t}\sin\omega t$, then the voltage equation is

$$Lpi + Ri + (1/Cp)i = E\varepsilon^{-\lambda t}\sin\omega t \,.\, (1)$$

or
$$i = EC\frac{p}{LCp^2 + RCp + 1} \,.\, \varepsilon^{-\lambda t}$$
$$\sin\omega t \,.\, (1) \quad . \quad . \quad (140)$$

By shifting, we get

$$i = EC\varepsilon^{-\lambda t}\frac{p - \lambda}{LC(p-\lambda)^2 + RC(p-\lambda) + 1}$$
$$\sin\omega t \,.\, (1) \quad . \quad . \quad (141)$$

Dropping the term $EC\varepsilon^{-\lambda t}$ for the time being, and applying the second form of the expansion theorem, we get

$$y = \frac{(j\omega - \lambda)\varepsilon^{-j\omega t}}{LC(j\omega - \lambda)^2 + RC(j\omega - \lambda) + 1} + \frac{(p_1 - \lambda)\varepsilon^{-p_1 t}}{2LC\omega_1(j\omega - p_1)}$$
$$- \frac{(p_2 - \lambda)\varepsilon^{-p_2 t}}{2LC\omega_1(j\omega - p_2)} \quad . \quad . \quad (142)$$

where $p_1 p_2 = \alpha \pm j\omega_1$, $\alpha = \lambda - R/2L$ and $\omega_1{}^2 = R^2/4L^2 - 1/LC$.

So, for the steady-state term—

$$y_s = \frac{-(\lambda \cos \omega t + \omega \sin \omega t) + j(\omega \cos \omega t - \lambda \sin \omega t)}{-\omega^2 LC - j\omega(2\lambda LC + RC) + (LC\lambda^2 - RC\lambda + 1)}$$

$$= \frac{-\sin(\omega t + \theta) + j\cos(\omega t + \theta)}{D\sqrt{(\lambda^2 + \omega^2)}} \quad . \quad . \quad . \quad . \quad (143)$$

where $\tan \theta = \lambda/\omega$.

On rationalizing and dropping imaginary terms we get

$$y_s = \frac{-\cos(\omega t + \theta) \, . \, [2\omega\lambda LC + RC\omega] - \sin(\omega t + \theta) \, [LC\lambda^2 - RC\lambda + 1 - \omega^2 LC]}{\sqrt{(\lambda^2 + \omega^2)}\,[(LC\lambda^2 - LC\omega^2 - RC\lambda + 1)^2 + \omega^2(2\lambda LC + RC)^2]}$$

With $\tan \phi = \dfrac{\omega C(2\lambda L + R)}{LC(\lambda^2 - \omega^2) - RC\lambda + 1}$ we get

$$y_s = -\frac{\sin(\omega t + \theta + \phi)}{\sqrt{(\lambda^2 + \omega^2)} \, . \, \sqrt{|D^2|}} \quad . \quad . \quad . \quad . \quad . \quad (144)$$

For the transient term we get, on substituting for p—

$$y_t = \frac{\varepsilon^{\alpha t}}{2jLC\omega_1}\left[\frac{\alpha - \lambda + j\omega_1}{j\omega - j\omega_1 - \alpha}\,\varepsilon^{-j\omega_1 t} - \frac{\alpha - \lambda - j\omega_1}{j\omega + j\omega_1 - \alpha}\,\varepsilon^{-j\omega_1 t}\right]$$

$$= \frac{\varepsilon^{\alpha t}}{j\omega_1 LC}\left[\frac{j(\alpha\lambda - \alpha^2 + j\alpha\omega - \omega_1^2)\sin \omega_1 t - \omega(\omega + j\lambda)\cos \omega_1 t}{\alpha^2 - 2j\omega\alpha - (\omega^2 - \omega_1^2)}\right]$$

Discarding imaginaries, we have

$$y_t = \frac{\varepsilon^{\alpha t}}{\omega_1 LC}\left[\frac{(\alpha^2 + \omega_1^2 - \omega^2)\,[\alpha(\lambda - \alpha) - \omega_1^2]\sin\omega_1 t + \omega_1[\lambda(\alpha^2 - \omega^2 + \omega_1^2) - 2\omega^2\alpha]\cos \omega_1 t}{[\alpha^2 + \omega_1^2 - \omega^2] + 4\alpha^2\omega^2}\right] \quad (145)$$

Then the complete solution is given by

$$i = EC\varepsilon^{-\lambda t}[y_s + y_t] \quad . \quad . \quad (146)$$

On neglecting damping, a usual procedure with radio circuits, we have $\omega_0^2 = 1/LC$ and $p = \lambda \pm j\omega_0$, and for the steady-state term, after some simplification, we get

$$y_s = -\frac{\omega_0^2}{\sqrt{(\omega^2 + \lambda^2)}} \cdot \frac{\sin(\omega t + \theta + \phi)}{\sqrt{[(\lambda^2 - \omega^2 + \omega_0^2)^2 + 4\lambda^2\omega^2]}} \quad . \quad (147)$$

The transient term becomes

$$y_t = \frac{\varepsilon^{\lambda t}}{2j\omega_0 LC}\left[\frac{j\omega_0\varepsilon^{j\omega_0 t}}{j(\omega-\omega_0)-\lambda} + \frac{j\omega_0\varepsilon^{-j\omega_0 t}}{j(\omega+\omega_0)-\lambda}\right]$$

$$\text{i.e. } y_t = \frac{\varepsilon^{\lambda t}}{LC}\left[\frac{\begin{array}{l}-(\lambda^2-\omega^2+\omega_0^2)\\ \quad[\omega_0\sin\omega_0 t+\lambda\cos\omega_0 t]-2\lambda\omega^2\cos\omega_0 t\end{array}}{(\lambda^2-\omega^2+\omega_0^2)^2+4\lambda^2\omega^2)}\right] \quad (148)$$

on rationalizing and discarding imaginaries.

If $\tan\theta_2 = \lambda/\omega_0$ and $\tan\phi_2 = 2\omega\lambda/(\lambda^2-\omega^2+\omega_0^2)$, then

$$y_t = \frac{\varepsilon^{\lambda t}}{LC}\left[-\frac{\cos\phi_2 \,.\, \sin(\omega_0 t+\theta_2)}{\sqrt{(\lambda^2+\omega_0^2)}} - \omega\sin\phi_2\cos\omega_0 t\right]\cdot\frac{1}{\Delta}$$

where $\Delta = \sqrt{[(\lambda^2-\omega^2+\omega_0^2)^2+4\lambda^2\omega^2]}$.

So the full expression for the current is

$$i = -\frac{EC\varepsilon^{-\lambda t}}{\Delta}\left\{\frac{\omega_0^2}{\sqrt{(\lambda^2+\omega^2)}}\sin(\omega t+\theta+\phi) + \frac{\varepsilon^{\lambda t}}{LC}\right.$$
$$\left.\left[\frac{\cos\phi_2 \,.\, \sin(\omega_0 t+\theta_2)}{\sqrt{(\lambda^2+\omega_0^2)}} - \omega\sin\phi_2\cos\omega_0 t\right]\right\} \quad . \quad (149)$$

6. **Theory of Ballistic Galvanometer.** Usually the action of the ballistic galvanometer is explained by considering the charge as given instantaneously to the galvanometer. We now investigate the movement when the charge is given over a period of time, as for example during the discharge of a condenser, when $e = E_0\varepsilon^{-t/CR} = E_0\varepsilon^{-\alpha t}$. From the theory of instrument movements, we get

$$mp^2\theta + \rho p\theta + \tau\theta = F \quad . \quad . \quad (150)$$

where ρ is the damping coefficient due to frictional effects and to the coil movement in the magnetic field. This latter effect may be expressed as G^2/r, where G is the galvanometer constant and r is the resistance of the coil circuit. The deflecting force

$$F = G\left(\frac{e-Lpi}{r}\right) \cong \frac{Ge}{R} = K\varepsilon^{-\alpha t}$$

$$\therefore\ \theta = K\frac{1}{mp^2+\rho p+\tau}\,.\,\varepsilon^{-\alpha t}\,.\,(1)$$

By shifting, we get

$$\theta = K\varepsilon^{-\alpha t}\,.\,\frac{1}{m(p-\alpha)^2+\rho(p-\alpha)+\tau}\,.\,(1) \quad . \quad . \quad (151)$$

Using the expansion theorem, we get

$$p = \alpha - \frac{\rho \mp \omega_0}{2m}$$

where $\omega_0 = \sqrt{(\rho^2 - 4m\tau)}$ and $Z'(p) = \pm\, 2m\omega_0$.

$$\therefore\ \theta = K\varepsilon^{-\alpha t}\left[\frac{1}{m\alpha^2 - \rho\alpha + \tau} + \frac{\varepsilon^{\alpha t}}{\omega_0}\left(\frac{\varepsilon^{-(\rho+\omega_0)t/2m}}{2m\alpha - \rho + \omega_0} - \frac{\varepsilon^{-(\rho-\omega_0)t/2m}}{2m\alpha - \rho - \omega_0}\right)\right] \quad . \quad (152)$$

The maximum deflection will occur at some time t_1, when $d\theta/dt = 0$, or

$$-\frac{\alpha}{m\alpha^2 - \rho\alpha + \tau}\varepsilon^{-\alpha t} - \frac{1}{\omega_0}\left\{\frac{[(\rho+\omega_0)/2m]\varepsilon^{-(\rho+\omega_0)t/2m}}{2m\alpha + \omega_0 - \rho} - \frac{[(\rho-\omega_0)/2m]\,\varepsilon^{-(\rho-\omega_0)t/2m}}{m\alpha - \omega_0 - \rho}\right\} = 0$$

so that

$$\theta_{max} = \frac{K}{2m\alpha\omega_0}\varepsilon^{-\rho t/2m}\left[\varepsilon^{\omega_0 t/2m} - \varepsilon^{-\omega_0 t/2m}\right] \quad . \quad (153)$$

For a critically damped instrument this reduces to

$$\theta_{max} = \frac{GE}{r\alpha}\,\frac{t}{2m^2}\,.\,\varepsilon^{-\rho t/2m} \quad . \quad . \quad . \quad . \quad (154)$$

7. **Operational Methods of Evaluating some Integrals.** The solution by the Duhamel method calls for evaluation of integrals of the form

$$y = \varepsilon^{-\alpha t}\int_{u=0}^{u=t} \varepsilon^{\alpha u}\sin \omega u\,.\,du$$

The integral term may be written down operationally* as

* See Appendix.

$$\frac{1}{p}\,.\,\varepsilon^{\alpha u}\,.\,\frac{\omega p}{p^2 + \omega^2}\,.\,(1)$$

Shifting $\varepsilon^{\alpha u}$ to the right, we get

$$\frac{1}{p}\,.\,\frac{\omega(p-\alpha)}{(p-\alpha)^2 + \omega^2}\,.\,\varepsilon^{\alpha t}$$

or $$\frac{1}{p}\,.\,\frac{\omega(p-\alpha)}{(p-\alpha)^2 + \omega^2}\,.\,\frac{p}{p-\alpha}\,.\,(1) = \frac{\omega}{(p-\alpha)^2 + \omega^2}\,.\,(1)$$

Evaluating this term by expansion, we get

$$y = \varepsilon^{-\alpha t}\left[\frac{\omega}{\alpha^2 + \omega^2} + \frac{\omega\varepsilon^{(\alpha + j\omega)t}}{2j\omega(\alpha + j\omega)} - \frac{\omega\varepsilon^{(\alpha - j\omega)t}}{2j\omega(\alpha - j\omega)}\right]$$

$$= \varepsilon^{-\alpha t}\left[\frac{\omega}{\alpha^2 + \omega^2} + \frac{\varepsilon^{\alpha t}}{\alpha^2 + \omega^2}(\alpha \sin \omega t - \omega \cos \omega t)\right]$$

Let $\tan \phi = \omega/\alpha$.

Then $$y = \frac{\sin(\omega t - \phi) + \varepsilon^{-\alpha t}\sin \phi}{\sqrt{(\alpha^2 + \omega^2)}}$$

If the expression is given as

$$y = \varepsilon^{-\alpha t}\int_{u=0}^{u=t} \varepsilon^{\alpha u} \sin(\omega u + \delta)du$$

then, proceeding as before, we get

$$\frac{1}{p} \cdot \varepsilon^{\alpha u} \cdot \frac{p\omega \cos \delta + p^2 \sin \delta}{p^2 + \omega^2} \cdot (1)$$

By shifting, we get

$$\frac{1}{p} \cdot \frac{(p - \alpha)\omega \cos \delta + (p - \alpha)^2 \sin \delta}{(p - \alpha)^2 + \omega^2} \cdot \frac{p}{p - \alpha} \cdot (1)$$

$$= \frac{\omega \cos \delta + (p - \alpha)\sin \delta}{(p - \alpha)^2 + \omega^2} \cdot (1)$$

By the expansion theorem we get

$$\cos \delta\left[\frac{\omega}{\alpha^2 + \omega^2} + \frac{\varepsilon^{\alpha t}}{\alpha^2 + \omega^2}(\alpha \sin \omega t - \omega \cos \omega t)\right]$$

$$+ \sin \delta\left[\frac{-\alpha}{\alpha^2 + \omega^2} + \frac{\varepsilon^{\alpha t}}{\alpha^2 + \omega^2}(\alpha \cos \omega t + \omega \sin \omega t)\right]$$

If $\tan \phi = \omega/\alpha$, then we have

$$y = \frac{1}{\sqrt{(\alpha^2 + \omega^2)}}[\sin(\omega t + \delta - \phi) - \varepsilon^{-\alpha t}\sin(\delta - \phi)]$$

Thus the operational method is of assistance in evaluating certain integrals. For further information on this method reference should be made to Heaviside.*

* O. HEAVISIDE: *Electro-magnetic Theory*, Vol. III, p. 234.

CHAPTER VI

TRANSIENT EFFECTS IN MACHINES

General. Machine work provides many illustrations of transient phenomena, but complete investigations are impossible, due to the complex behaviour of iron in the machine. However, if we neglect this effect in the interest of simplification, sufficient data are available to deal with the problems.

1. **Free Period of D.C. Armature: its Motional Capacity Effect.** In the case of a d.c. motor operating on constant voltage V we have that $V = E + Ri$, where E is the generated volts, R the resistance, and i the armature current. But we have

$$E = \phi Z \frac{\nu}{a} \cdot \frac{n}{60} \cdot 10^{-8}$$
$$= n\psi$$

where $\nu =$ no. of poles and $\psi = \phi Z \frac{\nu}{a} \cdot \frac{10^{-8}}{60} =$ the induction factor. If θ is the displacement of the armature in geometrical radians, then $Kd\theta/dt$ is the armature velocity in r.p.m. So we may write

$$E = K\psi \frac{d\theta}{dt}$$

Further, the torque $\propto Ei \propto b\psi i$.

So the voltage equation becomes

$$V = Ri + K\psi d\theta/dt \quad . \quad . \quad . \quad (155)$$

At light load the torque developed $=$ accelerating torque or

$$Jd^2\theta/dt^2 = b\psi i \quad . \quad . \quad . \quad (156)$$

where J is the mass moment of inertia of the armature.

As the voltage is constant, we get

$$Rpi + K\psi p^2\theta = pV = 0 \quad . \quad . \quad . \quad (157)$$

Substituting from equation (156), we have

$$Rpi + bK\psi^2 i/J = 0 \quad . \quad . \quad . \quad . \quad (158)$$

At $t = 0$, $i = V/R$, so we obtain

$$i = (V/R)\,\varepsilon^{-bK\psi^2 t/RJ} \quad . \quad (159)$$

Comparison of this equation with that for a condenser circuit, namely $i = (V/R)\,\varepsilon^{-t/CR}$ shows that the motional capacity effect of the armature is given by $J/bK\psi^2$.

When the armature inductance is taken into account, the following equations are obtained—

$$(R + Lp)i + K\psi p\theta = V \quad . \quad . \quad (160)$$

$$b\psi i = Jp^2\theta \quad . \quad . \quad (161)$$

or

$$i[Lp^2 + Rp + bK\psi^2/J] = 0 \quad . \quad . \quad (162)$$

When i is oscillatory, then $R^2/4L^2 < bK\psi^2/JL$, and

$$i = A\varepsilon^{-Rt/2Lt} \sin\left\{\left[\sqrt{\left(\frac{bK\psi^2}{JL} - \frac{R^2}{4L^2}\right)}\right]t - \phi\right\} \quad . \quad (163)$$

The frequency of oscillation is

$$\frac{1}{2\pi}\sqrt{\left(\frac{bK\psi^2}{JL} - \frac{R^2}{4L^2}\right)} \quad . \quad . \quad . \quad (164)$$

Neglecting the value of R as being very small, the frequency is $\frac{1}{2\pi}\sqrt{\left(\frac{bK\psi^2}{JL}\right)}$ and has a nominal value of about $\frac{1}{3}$ to $\frac{1}{10}$ cycle per second. The critical resistance below which the current oscillates is given by

$$R = 2\sqrt{(bK\psi^2L/J)}$$

2. **Stability of a Motor Operating on Fluctuating Voltage.** When the applied armature voltage is given by $V \sin \Omega t$, where Ω is the frequency of the armature voltage, we have

$$Lpi + Ri + K\psi p\theta = V \sin \Omega t \quad . \quad . \quad (165)$$

and

$$Jp^2\theta = b\psi i \quad . \quad . \quad (166)$$

By the above method we get

$$Lp^2i + Rpi + bK\psi^2 i/J = V\Omega \cos \Omega t \quad . \quad . \quad (167)$$

As we are interested in sustained oscillations only, we will consider the particular integral, namely—

$$i_s = V\frac{\cos(\Omega t + \theta)}{\sqrt{[R^2 + (bK\psi^2/J\Omega - L\Omega)^2]}} \quad . \quad (168)$$

and from equation (166) we have

$$p^2\theta = b\psi i/J$$

so that

$$\theta_s = -V\frac{b\psi \cos(\Omega t + \theta)}{\Omega^2\sqrt{[(RJ)^2 + (bK\psi^2/\Omega - LJ\Omega)^2]}} \quad . \quad (169)$$

It will be observed that θ has maximum values as follows—

1. With R small, when $\Omega^2 = Kb\psi^2/JL$.
2. With $R \cong \sqrt{[(2-\sqrt{3})\,bK\psi^2L/J]}$, when $\Omega = Kb\psi^2/JL\sqrt{3}$.
3. When $\Omega = 0$.

Also the value of i will be a maximum when $\Omega^2 = Kb\psi^2/JL$, i.e. when the frequency of disturbance is equal to the natural period of the machine, the value of the current is $(V/R)\cos\Omega t$.

3. Starting Conditions in D.C. Machines. It is sometimes found in practice that a motor is slow to pick up speed on starting. This is especially the case with compound-wound machines. In these it will be shown that the fields may act differentially during the starting period.

(i) SHUNT MOTOR. For a shunt machine we will assume that the air-gap flux is constant and that the field distortion is corrected, so that the back-e.m.f. depends on speed alone, i.e. $e = \psi n$.

The speed will be given by

$$n = \int_0^t a\,dt$$

where a is the acceleration = torque/inertia = $b\psi i/J$. So we have

$$e = (b\psi^2/J)\int_0^t i\,dt = K\int_0^t i\,dt$$

where $K = b\psi^2/J$.

Then the voltage equation is

$$Ri + Lpi + (K/p)i = V \qquad (170)$$

where R and L are the armature resistance and leakage inductance. Simplifying, and noting that at $t = 0$—

$$\left.\begin{aligned} i = I_0 = V/R &= c \\ di/dt = 0 \quad &= c \end{aligned}\right\} \qquad (171)$$

we get*

$$i = \frac{I_0[p^2 + (R/L)p]}{[p^2 + (R/L)p + (K/L)]}\cdot(1) \qquad (172)$$

i.e.

$$i = \frac{I_0}{2\beta}\varepsilon^{-\alpha t}[(\beta+\alpha)\varepsilon^{\beta t} + (\beta-\alpha)\,\varepsilon^{-\beta t} \qquad (173)$$

$$= I_0\,\varepsilon^{-\alpha t}[\cosh\beta t + (\alpha/\beta)\sinh\beta t] \qquad (174)$$

* Chapter IV, equation (97).

When the machine constants are such that $R^2/4L^2 < K/L$, then β is imaginary, and we get

$$i = I_0 \varepsilon^{-\alpha t}[\cos \beta t + \alpha/\beta \sin \beta t] \quad . \quad . \quad (175)$$

Now, $\alpha/\beta = \alpha/\sqrt{(\alpha^2 - K/L)} < 1$, so that the current after a time $\beta t = \pi/2$ is negative under these conditions, and the motor speed will "hang up" temporarily.

(ii) Compound Motor. For a compound-wound motor, the circuit may be represented in Fig. 19. Let the number of turns on the shunt field be an_1 and on the series field be n_1. Due to the series field, the air-gap flux is ci_1 and so the e.m.f.

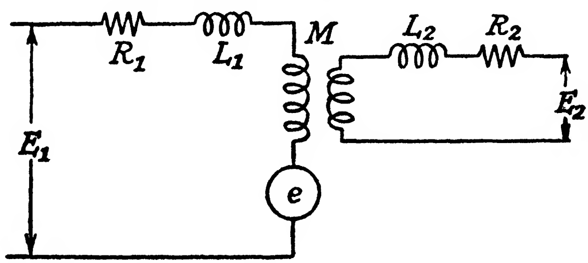

Fig. 19

induced in the series winding will be $p(ci_1)n_1 \times 10^{-8}$. Similarly the e.m.f. induced in the shunt field due to its flux aci_2 will be $p(aci_2)an_1 10^{-8}$. If e is the voltage generated in the armature by rotation, we get the following equations for the armature and series field—

$$E_1 = R_1 i_1 + L_1 p i_1 + p c i_1 n_1 10^{-8} + p a c i_2 n_1 10^{-8} + e$$
$$= R_1 i_1 + L_1 p i_1 + M(p i_1 + a p i_2) + e \quad . \quad . \quad . \quad (176)$$

For the shunt field—

$$E_2 = R_2 i_2 + L_2 p i_2 + p \,.\, a c i_2 a n_1 10^{-8} + p c i_1 a n_1 10^{-8}$$
$$= R_2 i_2 + L_2 p i_2 + M(a^2 p i_2 + a p i)_1 \quad . \quad . \quad . \quad (177)$$

Now, the value of the armature e.m.f. may be obtained from the same considerations as above. Here ϕ will be given by $c(i_1 + ai_2)$ and the torque by $b\phi i_1$, so that

$$e = K_1 \phi n = K_1 \phi \int_0^t \frac{b\phi i_1}{J} dt \quad . \quad . \quad (178)$$

The complete solution may be obtained from equations (176), (177), and (178). But it will be difficult to manipulate,

so we assume that ϕ is constant or $e = (K/p)i$, and the equations may be written thus—

$$E_1 = (R_1 + L_1p + Mp^2 + K/p)i_1 + aMpi_2 \quad . \quad . \quad (179)$$

$$E_2 = (R_2 + L_2p + a^2Mp)i_2 + aMpi_1 \quad . \quad . \quad . \quad (180)$$

From these equations, by eliminating i_1 and noting that $pE_1 = 0$ and $E_1 = E_2 = V$, we have

$$i_2 = V\left[\frac{(M + L)p^2 + R_1p + K}{Ap^3 + Bp^2 + Cp + KR_2}\right] . (1) \quad . \quad (181)$$

where $A = L_2M + a^2ML_1 + L_1L_2$

$B = MR_2 + L_1R_2 + a^2MR_1 + R_1L_2$

and $C = R_1R_2 + KL_2 + a^2MK$

As the coefficients A, B, and C are positive, the real roots are negative (imaginary roots, if they exist, must be in pairs and conjugate). Let the roots be given by $-\alpha$, $-\beta$, and $-\delta$, where α, β, and δ are positive. Substituting, we obtain

$$i_2 = E\left\{\frac{1}{R_2} - \sum_{(\alpha\beta\delta)}\left[\frac{Y(p)\varepsilon^{-pt}}{pdZ/dp}\right]\right\} \quad . \quad . \quad (182)$$

As R_2 is large, $1/R_2$ will be small, and the effect of the other three terms may be sufficient to cause the field current to be negative for small values of time.

In a similar manner the armature current may be formulated

as $$i_1 = -E\left[\frac{aMp^2}{Ap^3 + Bp^2 + Cp + KR_2}\right] . (1)$$

or $$i_1 = E\sum_{(\alpha\beta\delta)}\left[\frac{aMp^2\,\varepsilon^{-pt}}{pdZ/dp}\right] \quad . \quad . \quad . \quad . \quad (183)$$

For actual values of the currents i_1 and i_2 reference should be made to Ludwig's paper (see Bibliography).

Various methods have been devised to reduce the interaction of the fields, such as reducing the time constant of the shunt field and adding resistance to the series field. In certain cases, as in welding generators, the interaction between the fields is accentuated. On this matter reference should be made to papers by Miller.*

* Miller: *Trans. A.I.E.E.* (1934), Vol. 53, p. 1296; also Creedy: *Trans. A.I.E.E.* (1931), Vol. 50, p. 662.

4. Synchronous Machines: General Equations for Rotating Body

FREE OSCILLATIONS. For any rotating body the equation for free motion is given by

$$(J/g)\, p^2\theta + bp\theta + c\theta = 0 \qquad . \qquad . \quad (184)$$

where θ is the instantaneous value of angular oscillation measured in mechanical radians,
J is the amount of inertia in kg. m.^{2},
b is the damping force per unit angular velocity,
c is the control force per unit angular displacement.

This equation may be integrated as

$$\theta = A\varepsilon^{-\alpha t} \sin(\Omega t + \phi) \qquad . \qquad . \quad (185)$$

where α = damping constant = $bg/2J$,
Ω = natural angular velocity = $\sqrt{\left(\dfrac{cg}{J} - \dfrac{b^2g^2}{4J^2}\right)}$.

From this equation it is seen that if α is positive, then the oscillations die away exponentially and the system is stable. But if α is negative, then the amplitude of the oscillations increases and the system is unstable.

FORCED OSCILLATIONS. When fluctuations of the driving torque are considered, as in the case of a rotor driven by a reciprocating engine, then the equation becomes

$$(J/g)\, p^2\theta + bp\theta + c\theta = (T_{av} + T\cos\omega t)\,.\,(1) \quad . \quad (186)$$

where T_{av} is the mean torque,
T is the amplitude of the oscillating component superimposed on T_{av},
ω is the angular velocity of the disturbing force.

Generally, the oscillating torque curve would be analysed as a Fourier's series and formulated as $\sum\limits_{n=1}^{n=\infty} T_n \cos(n\omega t + \phi_n)$.

Here we concentrate on the simple form given above, and the additional deflection due to the varying component of torque is given by

$$\theta = T\left\{\frac{\cos(\omega t + \phi)}{\sqrt{[b^2\omega^2 + (J\omega^2/g - c)^2]}} + \frac{\varepsilon^{-\alpha t}}{J\Omega g} \frac{(\alpha^2 + \Omega^2)\cos(\Omega t + \psi)}{\sqrt{[(\alpha^2 + \Omega^2 - \omega^2) + 4\alpha^2\Omega^2]}}\right\} \quad . \quad (187)$$

where α and Ω are as defined in the immediately preceding part,

and $\tan \psi = \dfrac{(\alpha^2 + \Omega^2)^2 + \omega^2 (\alpha^2 + \Omega^2)}{2\alpha\gamma\Omega^2}$.

It is seen that the oscillatory motion consists of two frequencies given by $\omega/2\pi$ and $\Omega/2\pi$, the latter being more pronounced at the start and dying away after a time and leaving the forced oscillations. When the two are equal, we get a condition of mechanical resonance, and the resultant amplitude may be so large as to cause damage. When the forced oscillation is represented by a series expression, then there are more chances for resonance to occur. However, the amplitude is greatest at the gravest frequency, and it is usual to base design work on that frequency.

5. **Synchronous Motor.** We now apply the above to the case of a 2ν-pole synchronous motor which has a uniform reluctance for all positions of the armature. Let R and X_s denote the armature resistance and synchronous reactance per phase. These we consider constant. From the theory of a.c. machines we have the following developments for the synchronizing and damping torques, which give the values of c and b in the above torque equations.

(i) SYNCHRONIZING TORQUE. This is known to be proportional to the angle of displacement, and if the machine swings $\theta/2$ ahead of the mean position, then the net change of power per phase may be shown to be

$$\Delta P = (E^2X_s/Z_s^2) \sin(\theta/2) \cos(\theta/2) - EI \sin(\theta/2) \sin\phi \quad (188)$$

So the torque per unit space angle displacement from the mean position for an m-phase, 2ν-pole machine may be shown to be

$$c = \frac{m\nu^2 \Delta P}{(\theta/2) \,.\, 4\pi f} = \frac{m\nu^2}{g}\left[\frac{E^2L_s}{Z_s^2} - \frac{EI \sin\phi}{\omega}\right] \quad . \quad (189)$$

The first term is due to the circulating current, and the second is due to the change in phase angle between E and I. With $\cos\phi = 1$ and R negligible, we have

$$c = \frac{m\nu^2}{g\omega} \cdot \frac{E^2}{X_s} \quad . \quad . \quad . \quad (190)$$

(ii) THE DAMPING TORQUE. When the machine moves ahead of its mean position it will tend to give out more power and so

slow down. If it lags behind, it will tend to motor and so speed up. The forces set up will be proportional to speed, and so to $p\theta$. It may be shown that the power increment is given by

$$dP = EdI \cos\phi = (\phi_0{}^2/2)\,\omega d\omega\, R/Z^2 \quad . \quad . \quad . \quad (191)$$

where ϕ_0 is the max. value of the flux linkages. Also we have

$$dP = \text{torque} \times \text{space radians per sec} = T \,.\, d\overline{\omega}$$

So that torque per radian per second

$$b = \frac{dT}{d\omega} = \frac{\phi_0{}^2 R v^2 (R^2 - \omega^2 L^2)}{2g\,(R^2 + \omega^2 L^2)^2} \quad . \quad . \quad . \quad (192)$$

For an m-phase machine, when R is small this reduces to

$$b = -\,E^2 v^2 Rm/g\omega^4 L^2 \quad . \quad . \quad . \quad (193)$$

The negative value of b denotes that the machine is unstable, and the free oscillations will increase indefinitely. Insertion of damping windings in the pole faces ensures that b is positive. This introduces non-symmetry, and the problem becomes unwieldy.*

In the case of forced oscillations discussed in Section 4 above, the values of b and c will change with change of speed and of displacement respectively. The synchronizing torque will consist of two components, one due to steady displacement and the other due to the oscillating displacement. To determine the final value of the synchronizing torque a graphical method due to Rosenberg† is used. Illustrations of this may be found in texts on electrical machine design.‡

6. **Parallel Operation of A.C. Generators: Self-excited Vibrations.** A problem which is closely allied to that of synchronous motor operation dealt with in the last section is that of parallel operation of a.c. generators. Here the successful operation of machines in parallel depends on the characteristics of both the alternator and the turbine, or in other words we must take into account the governor stability as well as the alternator stability.

For the governor, we adopt the following approximate force equation for free vibration—

$$mp^2x + c_1px + k_1x = 0 \quad . \quad . \quad . \quad (194)$$

* See e.g. PRESCOTT AND RICHARDSON: *Journ. I.E.E.* (1934), Vol. 75, p. 497.

† ROSENBERG: *E.T.Z.* (1902), Vol. 23, p. 450.

‡ E.g. MILES WALKER: *Specification and Design of Dynamo-Electric Machinery* (Longmans).

where x is the movement of the governor sleeve,

m is the equivalent mass of the rotating parts of the governor referred to the motion of the sleeve,

c_1 is the damping force constant of the governor due to the oil dashpot,

k is the restoring force per unit displacement of the sleeve due to the spring.

For the alternator we have, under similar conditions, that

$$Jp^2\theta + c_2p\theta + k_2\theta = 0 \qquad . \quad . \quad (195)$$

where θ is the displacement in mechanical radians of the rotor,

J is the mass moment of inertia of all the generator and turbine rotating parts,

c_2 is the damping constant due to the extra power torque developed and to the braking action of the damping winding,

k_2 is the synchronizing torque of the generator.

As we are going to combine both sets we must measure all quantities in the same units, say in lb.-sec. units.

In the complete set the equations (194) and (195) may be combined. So we have certain relationships between the various quantities. For a small speed change ($\Delta\omega$) let the movement of the governor sleeve be x. This movement will open the steam valve, and so the engine torque will be increased proportionately to x by ΔT, say. So now we have $\Delta T = kx$, where k is a constant. But due to sleeve movement, we immediately have a restoring force on the governor amounting to $- c\Delta\omega = - cp\theta$, so that our equations for disturbed motion become

$$mp^2x + c_1px + k_1x = - c\Delta\omega = - cp\theta \qquad . \quad (196)$$

and

$$Jp^2\theta + c_2p\theta + k_2\theta = kx \quad . \quad . \quad . \quad . \quad (197)$$

and we get

$$[(Jp^2 + c_2p + k_2)\,(mp^2 + c_1p + k_1) + kcp]\,\theta = 0 \quad . \quad (198)$$

From this equation we have the determinantal equation

$$p^4 + a_1p^3 + a_2p^2 + a_3p + a_4 = 0 \quad . \quad . \quad (199)$$

where

$$\left.\begin{aligned} a_1 &= c_1/m + c_2/J\,;\ a_2 = k_1/m + k_2/J + c_1c_2/Jm \\ a_3 &= (c_1k_2 + c_2k_1 + ck)/Jm\,;\ a_4 = k_1k_2/Jm \end{aligned}\right\}. \quad (200)$$

Now we may introduce constants derived from the consideration of each unit working independently as follows—

(*a*) Damping coefficients $\alpha_1 = - c_1/2m$; $\alpha_2 = - c_2/2J$.

(*b*) Natural angular velocities $\omega_1^2 = k_1/m$; $\omega_2^2 = k_2/J$.

It is seen that all coefficients in our determinantal equation are positive; further, we obtain by reference to Section 9, Chapter I, equation (16) as the criterion of stability, that $a_1a_2a_3 - a_3^2 - a_1^2a_4$ must be positive, or, in the limit,

$$a_1a_2a_3 = a_3^2 + a_1^2a_4 \qquad . \quad . \quad . \ (201)$$

We may write down our coefficients as—

$$a_1 = - 2(\alpha_1 + \alpha_2);\ a_2 = \omega_1^2 + \omega_2^2 + 4\alpha_1\alpha_2$$
$$a_3 = - 2\alpha_2\omega_1^2 - 2\alpha_1\omega_2^2 + 2\gamma;\ \ a_4 = \omega_1^2\omega_2^2$$

where $\gamma = ck/2Jm$.

Now, suppose $\hat{x}$ is the maximum sleeve travel and $\Delta\omega$ is the maximum change of speed, then we have

$$c\Delta\omega = k_1\hat{x}$$

or

$$c = k_1(\hat{x}/\Delta\hat{\omega})$$

Denoting $\Delta\omega/\omega$ by δ, the governor regulation, and noting that the normal torque, T_n, is given when the sleeve movement is $\hat{x}$, then we have

$$k\hat{x} = T_n$$

So that

$$\gamma = \frac{k}{2Jm} \cdot \frac{k_1\hat{x}}{\omega(\Delta\omega/\omega)} = \frac{T_n}{2\delta} \cdot \frac{k_1}{J\omega m}$$
$$= \omega_1^2T_n/2J\omega\delta = \omega_1^2/2t\delta$$

where $t = J\omega/T_n$ is the time to bring the set up to full speed from standstill.

Substituting these relationships in equation (201), we get

$$(\alpha_1 + \alpha_2)[\omega_2^2(1 + \rho^2) + 4\alpha_1\alpha_2]\,[\alpha_2 + \alpha_1/\rho^2 - 1/2\delta t]$$
$$- \omega_2^2(\alpha_1 + \alpha_2)^2 - \rho^2\omega_2^2[\alpha_2 + \alpha_1/\rho^2 - 1/2\delta t]^2 = 0 \quad (202)$$

where $\rho = \omega_1/\omega_2$.

From this equation it may be deduced that—

(*a*) When the governor natural frequency exceeds the generator natural frequency, then damping of the governor is necessary.

(*b*) When the governor natural frequency is less than that of the alternator, then damping windings are necessary.

(*c*) For isosynchronism, damping is necessary in both governor and alternator.

7. **Pulling into Step of a Synchronous Motor.** Here we have an interesting but insoluble problem. In starting a synchronous motor the usual procedure is to supply the stator winding with polyphase currents; the short-circuited damping winding on field system acts as the rotor winding of an induction motor. The motor thus runs up to a subsynchronous speed. In this condition the field poles will slip backwards through the synchronously rotating stator field. When the field system is energized, the two fields will eventually lock together and the rotor will then move forward at synchronous speed.

The phenomenon which occurs between the induction-motor and synchronous-motor stages is exceedingly complex, and the following points are of importance—

(*a*) Due to the damping winding being asymmetrical, the induction-motor torque will be uneven.

(*b*) With a salient-pole machine the reluctance of the air gap varies considerably, and will aggravate the non-uniformity of the torque under (*a*).

(*c*) When the main field switch is closed, then the field current will build up according to the law $I(1 - \varepsilon^{-\alpha t})$. The rate will depend on the time constant of the field, and a period of time will elapse before the full synchronizing torque is experienced by a motor.

With these points in mind, we can enumerate the torques that will be experienced and must be considered for any pulling-into-step operation.

(*a*) The amount of the shaft load torque: the greater the load, the longer the time that will be required to accelerate it and the greater will be the slip of the induction motor.

(*b*) The torque developed as an induction motor and the characteristic (torque/slip) curve: under this heading the effect of the unbalanced windings and reluctance must be considered.

(*c*) The torque developed by the motor when the field poles are excited: this will depend on the angle (θ) between the excitation and armature magnetic axes at the moment of the closure of the switch.

Thus we get an equation of the type

$$P_1 p^2\theta + P_2(1 - b\cos 2\theta)p\theta + P_3 \sin 2\theta + P_4 \sin\theta \,.\, (1) = P_5 \qquad (203)$$

where $P_1 p^2\theta$ is the inertia power and is equal to $18{\cdot}6(Wr^2)\,(f^2/v^2)$

$\times\ 10^{-6}$ kW. per electrical degree per sec^2. $\times\ (d^2\theta/dt^2)$ and where Wr^2 = inertia in lb. ft.^{2},

f = frequency,

ν = number of poles.

The induction-motor power $P_2 = P_5/360fs$ kW./electrical degree/sec., and $(1 - b\cos 2\theta)$ is a term to take into account the asymmetrical damper windings. $P_3 \sin 2\theta$ is the reluctance torque. The value of P_3 is given by Doherty and Nickle* as $\frac{3(x_d - x_q)}{2x_dx_q} \cdot \frac{V^2}{1\ 000}$ kW., where x_d and x_q are the reactances in the direct and quadrature axes respectively. $P_4 \sin\theta \,.\, (1)$ is the synchronizing effect of the field current. The unit function sign is used to denote that before the field is applied this term is zero.

The solution of these equations cannot be derived by ordinary mathematical means, and recourse is had to an integraph.† This method has been applied to the solution of various problems.‡

From the results obtained in the two papers quoted below for the limiting values of slip for successful pulling into step we note—

(*a*) For a cylindrical rotor

$$s = \frac{620}{n}\sqrt{\left[\frac{P_4}{f(Wr^2)}\right]} \qquad . \quad . \quad (204)$$

(*b*) For a salient-pole motor

$$s = \frac{590}{n}\sqrt{\left[\frac{P_4}{f(Wr^2)}\right]} \qquad . \quad . \quad (205)$$

8. **Damping Action due to a Shaded Core.** In contactor and relay work, the iron core is shaded by a close-fitting copper ring. In the case of synchronous machines there is the allied problem of the short-circuited damping windings. Due to the analogy existing, we will here consider the first case, and to simplify our problem we assume a laminated core so that eddy currents are negligible.

* DOHERTY AND NICKLE: *Trans. A.I.E.E.* (1926), Vol. 45, p. 912.

† BUSH, GAGE, AND STEWART: "A Continuous Integraph," J. Franklin Inst. (1927), Vol. 203, p. 63.

‡ E.g. EDGERTON AND FOURMARIER: "Pulling into Step of Salient-pole Machines," *Trans. A.I.E.E.* (1931), Vol. 50, p. 769; EDGERTON AND ZAK: "Pulling into Step of a Synchronous Induction Motor." *Journ. I.E.E.* (1930), Vol. 68, p. 1205 .

Consider an exciting winding of n turns on a core arranged as in Fig. 20. Then

$$V = Ri + n\,d\phi/dt \quad . \quad (206)$$

with ϕ expressed in 10^2 megalines.

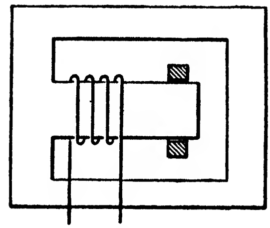

FIG. 20

The current set up in the solid copper ring will be

$$i_s = (1/R_s)\,d\phi/dt\,. \quad . \quad (207)$$

where R_s = resistance of the copper ring.

Now, the m.m.f. in the core will be the vector sum of that due to the two windings: we may write

$$\phi = K(ni - i_s)$$

where K is the flux per ampere-turn

and $\qquad Kni = \phi + Ki_s = [1 + (K/R_s)p]\phi$

So that $\qquad i = (1/Kn)\,[1 + (K/R_s)p]\phi$

Substituting in (206), we get

$$V = (R/Kn)\phi + (R/R_s n)p\phi + np\phi \quad . \quad (208)$$

Denoting KnV/R by ϕ_s, the steady-state value of the flux, we obtain

$$\phi_s = [1 + (K/R_s)p + (Kn^2/R)p]\phi$$

Hence we obtain by expansion that

$$\phi = \phi_s\{1 - \exp.\,[-t/(K/R_s + Kn^2/R)]\} \qquad (209)$$

When the copper ring is absent we have

$$\phi = \phi_s\{1 - \exp.\,[-t/(Kn^2/R)]\} \quad . \quad . \quad (210)$$

It is seen thus that the presence of the short-circuit ring causes the time lag of the flux to be increased. The increase depends on the resistance of the ring and on the square of the number of turns in the exciting coil. In any machine equipped with short-circuited windings the flux will be tardy in either the building-up or the decaying processes.

9. **Short-circuit Current of an Alternator: Symmetrical Fault.** The calculation of the transient short-circuit current in a.c. machines has been the subject of several papers, each author eliminating one or more of the assumptions of his predecessor. Naturally, as such work proceeds the mathematics become more cumbersome. We may start with the elementary discussion given in Section 4, Chapter V, wherein we assumed, among other things, no variation in the field current during the short

circuit. We are able to remove this error by a method developed by Ku.* Here we assume that the stator windings are symmetrical, the rotor speed constant, and the resistances and inductances of the stator winding constant and independent of rotor position and of current; also that the mutual effect between rotor and stator depends on the relative displacement angle; and that the machine is not saturated and the harmonics in the distribution of the air-gap flux are negligible. This is very close to the case of the machine with a cylindrical rotor.

When a three-phase short circuit occurs on a machine, the phase voltages v_1, v_2, v_3 become zero. So we may consider that they are annulled by equal and opposite voltages at short circuit. The current may thus be assumed to be the steady-state current at the moment of short circuit plus the transient current set up by application of these annulling voltages at the terminals of the machine.

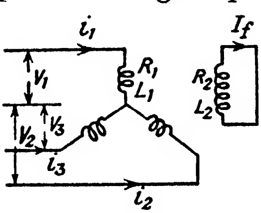

FIG. 21

Let the currents, etc., be shown as in Fig. 21. Then we have the following equations for suddenly applied terminal e.m.f.s

$$(R_1 + L_1p)i_1 + Mpi_f \cos\theta = v_1 \quad . \quad (211)$$

$$(R_1 + L_1p)i_2 + Mpi_f \cos(\theta - 2\pi/3) = v_2 \quad . \quad (212)$$

$$(R_1 + L_1p)i_3 + Mpi_f \cos(\theta + 2\pi/3) = v_3 \quad . \quad (213)$$

$$(R_2 + L_2p)i_f + Mp[i_1 \cos\theta + i_2 \cos(\theta - 2\pi/3) + i_3 \cos(\theta + 2\pi/3)] = 0 \quad . \quad (214)$$

Here L_1 = synchronous self-inductance of a stator phase plus the mutual effect of the other stator phases.

As we are considering a symmetrical three-phase short circuit without neutral connection, we may write

$$i_1 + i_2 + i_3 = 0$$

$$v_1 + v_2 + v_3 = 0$$

So equation (214) becomes

$$(R_2 + L_2p)i_f + \sqrt{3}Mp[i_1 \cos(\theta - \pi/6) + i_2 \cos(\theta - \pi/2)] = 0 \quad (215)$$

Further, we know that

$$i_1 = I\cos\omega t = (I/2)[\varepsilon^{j\omega t} + \varepsilon^{-j\omega t}] = \overset{\rightarrow}{\imath} + \overset{\leftarrow}{\imath}, \text{ say.}$$

* KU: *Trans. A.I.E.E.* (1929), Vol. 48, p. 707.

and $\quad i_2 = I\cos(\omega t - 2\pi/3) = (I/2)[\varepsilon^{j(\omega t - 2\pi/3)} + \varepsilon^{-j(\omega t - 2\pi/3)}]$

$$= \hat{\imath}\,\varepsilon^{-j2\pi/3} + \check{\imath}\,\varepsilon^{+j2\pi/3}$$

$$= \hat{\imath}a^2 + \check{\imath}a, \text{ say.}$$

On substituting these in equation (211) and equating like components of current and voltage, we have

$$(R_1 + L_1p)\hat{\imath} + \tfrac{1}{2}Mpi_f\varepsilon^{j\theta} = \hat{v} \quad . \quad . \quad (216)$$

and

$$(R_1 + L_1p)\check{\imath} + \tfrac{1}{2}Mpi_f\varepsilon^{-j\theta} = \check{v} \quad . \quad . \quad (217)$$

Equation (215) now becomes

$$(R_2 + L_2p)i_f + \tfrac{3}{2}Mp(\check{\imath}\varepsilon^{-j\theta} + \check{\imath}\varepsilon\ \ ) = 0 \qquad (218)$$

If we assume that the short circuit occurs when the excitation axis coincides with that of phase I, then $\theta = \omega t$. On multiplying equation (216) by $\varepsilon^{-j\omega t}$ and (217) by $\varepsilon^{j\omega t}$ and using the shifting theorem, we get

$$[R_1 + L_1(p + j\omega)]\hat{\imath}\varepsilon^{-j\omega t} + \tfrac{1}{2}M[p + j\omega]i_f = \hat{v}\varepsilon^{-j\omega t} \qquad (219)$$

$$[R_1 + L_1(p - j\omega)]\check{\imath}\varepsilon^{j\omega t} + \tfrac{1}{2}M[p - j\omega]i_f = \check{v}\varepsilon^{j\omega t} \quad . \qquad (220)$$

From equations (218), (219), and (220) the values of $\check{\imath}$ and $\hat{\imath}$ are obtained, as follows—

$$\hat{\imath} = \frac{1}{Z(p)} \cdot \frac{1}{L_1\sigma}\left[\left(p + \frac{R_2}{L_2}\right)(p + R_1/L_1 - j\omega)\hat{v}\right.$$
$$\left. - \frac{3}{4}\frac{M^2}{L_1L_2}\,p(p - j\omega)\hat{v} + \frac{3}{4}\frac{M^2}{L_1L_2}\,p(p + j\omega)v\,\varepsilon^{2j\omega t}\right]$$

and $\check{\imath}$ is given by a similar expression with $\check{v}$ substituted for $\hat{v}$ and $j\omega$ for $-j\omega$, i.e. it is a conjugate expression.

Here $\sigma = (1 - 3M^2/2L_1L_2)$ and

$$Z(p) = p^3 + \left[\frac{R_1}{L_1} + \frac{1}{\sigma}\left(\frac{R_1}{L_1} + \frac{R_2}{L_2}\right)\right]p^2 +$$
$$\left[\frac{1}{\sigma}\frac{R_1}{L_1}\left(\frac{R_1}{L_1} + \frac{2R_2}{L_2}\right) - \omega^2\right]p + \frac{1}{\sigma}\frac{R_2}{L_2}\left(\frac{R_1^2}{L_1^2} - \omega^2\right)$$

From $Z(p)$ the roots are obtained in the usual way. With the normal machine constants, one root is real and the other two complex.

When the voltage v is given by $V\sin\omega t = (V/2j)\,[\varepsilon^{j\omega t} - \varepsilon^{-j\omega t}]$

Then $\qquad \hat{v} = (V/2j)\varepsilon^{j\omega t}$ and $\check{v} = -(V/2j)\varepsilon^{-j\omega t}$

The values of the currents are now given by

$$\hat{\imath} = \frac{V}{2j}\varepsilon^{j\omega t}\frac{1}{\sigma L_1 Z(p)}\left[\left(p + \frac{R_2}{L_2}\right)(p + R_1/L_1 - j\omega) - \frac{3}{2}\cdot\frac{M^2}{L_1L_2}p^2\right]$$

$$\check{\imath} = \frac{V}{2j}\varepsilon^{-j\omega t}\frac{1}{\sigma L_1 \,.\, Z(p)}\left[\left(p + \frac{R_2}{L_2}\right)\left(p + \frac{R_1}{L_1} + j\omega\right) - \frac{3}{2}\cdot\frac{M^2}{L_1L_2}p^2\right]$$

As these currents are conjugates, we have

$$i = \hat{\imath} + \check{\imath} = 2\hat{\imath} \text{ (real)}$$

When the short circuit occurs under load conditions there is an angle α between the excitation e.m.f. and the terminal volts. On taking account of the time (θ) at which the switch is closed, the terminal voltage will be fully specified by $V\varepsilon^{j(\omega t+\theta+\alpha)}$. As further analysis is difficult, we will now use the results given in Ku's paper to show the effect of short-circuiting the machine at no load but with $\theta = -71{\cdot}3°$.

The machine constants are as follows—

$$\begin{array}{lll}
R_1 = 0{\cdot}0232 & R_2 = 0{\cdot}138 & M = 8{\cdot}34 \times 10^{-3} \\
L_1 = 5{\cdot}66 \times 10^{-3} & L_2 = 19{\cdot}3 \times 10^{-3} & \omega = 377 \\
\sigma = 0{\cdot}046 & R_1/L_1 = 4{\cdot}1 & R_2/L_2 = 7{\cdot}13 \\
1/\sigma L_1 = 3\,840 & V_{max} = 188 &
\end{array}$$

$$Z(p) = p^3 + 248p^2 + 144 \times 10^3 p + 220 \times 10^5$$

The roots are $p_1 = -169$; $p_2p_3 = -39{\cdot}6 \pm j359$

With $p_1 = -169$, the contribution is found to be

$$\hat{\imath}_{p_1} = \frac{188}{2}\varepsilon^{j(377t-71°-90°)} \times \frac{3\,840 \times 6\,100\,\underline{|89{\cdot}5°}}{-246 \times 10^6}\varepsilon^{-169t}$$

$$= 894\,\varepsilon^{-169t}\,\varepsilon^{j(377t+109°)}$$

Twice the real part gives the total contribution as

$$1\,790\varepsilon^{-169t}\cos(377t + 109°)$$

The value of the total current is given as

$$i = 1\,790\varepsilon^{-169t}\cos(377t + 109°) + 949\varepsilon^{-39{\cdot}6t}\cos(736t - 51°) + 1\,038\varepsilon^{-39{\cdot}6t}\cos(18t - 88{\cdot}7°) + 88\cos(377t + 109°)$$

The last term is the sustained value, and it will be noted that the largest term is attenuated the most. The term of double frequency is not so strongly damped, and there is a further term of about zero frequency.

The expression for the field current under no-load conditions may be shown to be

$$i_f = \frac{3}{2} \cdot \frac{M}{L_1 L_2} \frac{V}{\sigma} \cdot \frac{\omega p}{Z(p)}$$

In this case it is given by

$$i_f = 94 \times 10^4 \times 188 \times \frac{p}{Z(p)}$$

With the value of the roots as given we have that the contribution of p_1 is

$$i_{f_1} = 94 \times 10^4 \times 188 \times \frac{-169}{-24{\cdot}6 \times 10^6} \times \varepsilon^{-169t}$$
$$= 1\,210\varepsilon^{-169t}$$

On evaluating the contributions of the other roots, we have

$$i_f = 1\,210\varepsilon^{-169t} + 1\,290\varepsilon^{-39{\cdot}6t} \cos(359t - 160°)$$

i.e. a current of practically fundamental frequency superimposed on an exponential curve.

Of recent date several contributions towards the solution of unbalanced faults have been made. As these are difficult to summarize, we append some references dealing with this subject.

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CHAPTER VII

PROPAGATION OF ELECTRIC WAVES ALONG A LINE

1. **Steady-state Solution.** A transmission line may be regarded as composed of sections: the greater the number of sections, the closer is the representation. In the limit, with sections of length dx, the representation is complete, and solutions obtained under these conditions are designated as *smooth* line solutions. The arrangement of a section is shown in Fig. 22,

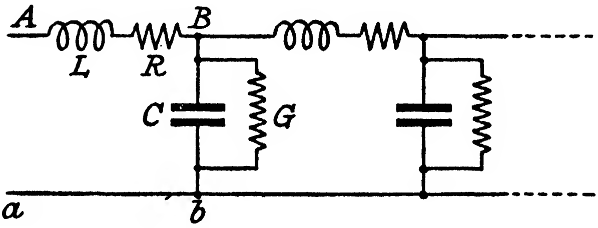

FIG. 22

where R is the resistance, L the inductance, C the capacity, and G the insulation conductance, all per unit length of the line. In an elementary way, the flow of current may be considered to occur as follows—

Due to the impressed voltage at Aa, a current will flow in the section $ABba$. The condenser element will assume a

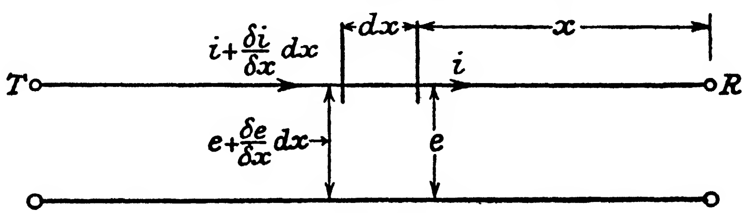

FIG. 23

charge, and a voltage will be impressed on the next section, with a resultant current therein. Repetition of this process eventually gives a voltage at the far end of the line. The initial diffusion of electricity through the wire is termed the building-up process, and the final stage is the steady state. As the theory concerning steady-state conditions is well known

to all power engineers, we will give a brief summary and develop therefrom the theory concerning the process of building up.

For the section given in Fig. 23, let the voltages and currents be as shown. Then

$$e + \frac{\partial e}{\partial x} dx = e + i(Rdx) + (Ldx)pi \quad . \quad . \quad . \quad (221)$$

and $$i + \frac{\partial i}{\partial x} dx = i + e(Gdx) + (Cdx)pe \quad . \quad . \quad . \quad (222)$$

where x is measured from the receiving end of the line.

These equations reduce to

$$\frac{\partial e}{\partial x} = iR + Lpi$$

and $$\frac{\partial i}{\partial x} = eG + Cpe$$

for harmonic quantities the potency of $p = j\omega$. So we have

$$\frac{\partial e}{dx} = i[R + j\omega L] = iZ \quad . \quad . \quad . \quad . \quad (223)$$

and $$\frac{\partial i}{\partial x} = e[G + j\omega C] = eY \quad . \quad . \quad . \quad . \quad (224)$$

By differentiation we have

$$\frac{\partial^2 e}{\partial x^2} = Z\frac{\partial i}{\partial x} = ZYe = \alpha^2 e \quad . \quad . \quad . \quad (225)$$

and $$\frac{\partial^2 i}{\partial x^2} = Y\frac{\partial e}{\partial x} = ZYi = \alpha^2 i \quad . \quad . \quad . \quad (226)$$

where $$\alpha = \sqrt{(ZY)} = \sqrt{[(R + j\omega L)(G + j\omega C)]} \quad (227)$$

A solution of these equations is

$$E_x = A\varepsilon^{\alpha x} + B\varepsilon^{-\alpha x} \quad . \quad . \quad . \quad . \quad (228)$$

and $$I_x = (\alpha/Z)[A\varepsilon^{\alpha x} - B\varepsilon^{-\alpha x}] \quad . \quad . \quad . \quad (229)$$

where A and B are constants, determinate from terminal conditions.

For example, at $x = 0$ let E_R, I_R be the receiver volts and current respectively. On substitution we get

$$A = \tfrac{1}{2}[E_R + (Z/\alpha)I_R] \quad \text{and} \quad B = \tfrac{1}{2}[E_R - (Z/\alpha)I_R]$$

So the voltage and current are given by

$$E_x = \tfrac{1}{2}[E_R + (Z/\alpha)I_R]\varepsilon^{\alpha x} + \tfrac{1}{2}[E_R - (Z/\alpha)I_R]\varepsilon^{-\alpha x} \quad . \quad (230)$$

$$= E_R[\varepsilon^{\alpha x} + \varepsilon^{-\alpha x}]/2 + (Z/\alpha)I_R[\varepsilon^{\alpha x} - \varepsilon^{-\alpha x}]/2$$

$$= E_R \cosh \alpha x + I_R\sqrt{(Z/Y)} \,.\, \sinh \alpha x \quad . \quad . \quad (231)$$

also

$$I_x = \tfrac{1}{2}\sqrt{(Y/Z)}\{[E_R + I_R\sqrt{(Z/Y)}] \,.\, \varepsilon^{\alpha x} - [E_R - I_R\sqrt{(Z/Y)}] \,.\, \varepsilon^{-\alpha x}\} \quad . \quad . \quad . \quad (232)$$

$$= I_R \cosh \alpha x + E_R\sqrt{(Y/Z)} \,.\, \sinh \alpha x \quad . \quad . \quad (233)$$

If the voltage and current at the transmitter ($E_T I_T$) had been used in the determination of the constants, x measured from the receiving end, we have

$$E_x = E_T \cosh \alpha(l - x) - I_T\sqrt{(Z/Y)} \,.\, \sinh \alpha(l - x) \quad . \quad (234)$$

$$I_x = I_T \cosh \alpha(l - x) - E_T\sqrt{(Y/Z)} \,.\, \sinh \alpha(l - x) \quad . \quad (235)$$

When x is measured from the transmitting end we get

$$E_x = E_T \cosh \alpha x - I_T\sqrt{(Z/Y)} \,.\, \sinh \alpha x \quad . \quad (236)$$

$$I_x = I_T \cosh \alpha x - E_T\sqrt{(Y/Z)} \,.\, \sinh \alpha x \quad . \quad (237)$$

When the far end of the line is connected to a load, let $E_R/I_R = \sigma$, the load impedance. Then we have, for equations (230) and (233)

$$E_x = E_R\left[\cosh \alpha x + \frac{1}{\sigma}\sqrt{\left(\frac{Z}{Y}\right)}\sinh \alpha x\right]$$

and

$$I_x = I_R\left[\cosh \alpha x + \sigma\sqrt{\left(\frac{Y}{Z}\right)}\sinh \alpha x\right]$$

Denoting $\sigma\sqrt{(Y/Z)}$ by $\tanh \theta$, these equations become

$$E_x = E_R \frac{\sinh(\alpha x + \theta)}{\sinh \theta} \quad . \quad . \quad . \quad (238)$$

and

$$I_x = I_R \frac{\cosh(\alpha x + \theta)}{\cosh \theta} \quad . \quad . \quad . \quad (239)$$

So then

$$E_T/I_T = \sigma \tanh(\alpha l + \theta) \,.\, \coth \theta$$

$$= \sqrt{(Z/Y)} \,.\, \tanh(\alpha l + \theta)$$

$$= Z_0 \tanh(\alpha l + \theta) \quad . \quad . \quad . \quad (240)$$

With these results, the following relationships for various terminal conditions may be obtained—

Line Termination	σ	E_x	I_x	Z_x
Earthed . .	0	$E_T \dfrac{\sinh \alpha x}{\sinh \alpha l}$	$I_T \dfrac{\cosh \alpha x}{\cosh \alpha l}$	$\sqrt{(Y/Z)} \, . \tanh \alpha x$
Open . . .	∞	$E_T \dfrac{\cosh \alpha x}{\cosh \alpha l}$	$I_T \dfrac{\sinh \alpha x}{\sinh \alpha l}$	$\sqrt{(Y/Z)} \, . \coth \alpha x$
Infinite Line. .	$\sqrt{(Z/Y)}$	$E_T \varepsilon^{-\alpha(l-x)}$	$I_T \varepsilon^{-\alpha(l-x)}$	$\sqrt{(Z/Y)} = \sigma$

2. **Roots of α.** The value of α is given by

$$\sqrt{[(R + Lp)\,(G + Cp)]}$$

or for harmonic quantities with $p \doteq j\omega$—

$$\begin{aligned} \alpha &= \sqrt{[(RG - \omega^2 LC) + j\omega(LG + RC)]} \\ &= \beta \pm j\gamma, \text{ say} \end{aligned}$$

The values of β and γ are obtained by squaring both values of α and equating the reals and imaginaries. The following results are obtained—

$$\beta = \sqrt{[\tfrac{1}{2}(|Z| \, . \, |Y| + RG - \omega^2 LC)]} . \qquad . \quad (241)$$

$$\gamma = \sqrt{[\tfrac{1}{2}(|Z| \, . \, |Y| - RG + \omega^2 LC)]} . \qquad . \quad (242)$$

where $|Z| = \sqrt{(R^2 + \omega^2 L^2)}$ and $|Y| = \sqrt{(G^2 + \omega^2 C^2)}$.

By differentiating the expression for β, with respect to ω, it may be shown that β is independent of frequency when $R/L = G/C$ and α has a value of $\sqrt{(RG)}$. Under like conditions $\gamma = \omega\sqrt{(LC)}$.

The physical meaning of these expressions may be illustrated by considering an infinite line. With x measured from the transmitter, we have

$$E_x = E_T \varepsilon^{-\alpha x} = E_T \varepsilon^{-(\beta + j\gamma)x}$$

Now, $\varepsilon^{-j\gamma x} = \cos \gamma x - j \sin \gamma x$. So we have

$$E_x = E_T \varepsilon^{-\beta x}[\cos \gamma x - j \sin \gamma x]$$

Thus an applied voltage E_T is attenuated to $E_T \varepsilon^{-\beta x}$ due to passage along a line of length x. In addition, it suffers a phase change of γx. Thus, when $\gamma x = 2\pi$ or $x = 2\pi/\gamma$ miles, the current and voltage at x are in phase with the sending current

and volts; but when $x\gamma = \pi$ or $x = \pi/\gamma$, these quantities are in phase opposition. A spiral diagram shows this idea clearly.

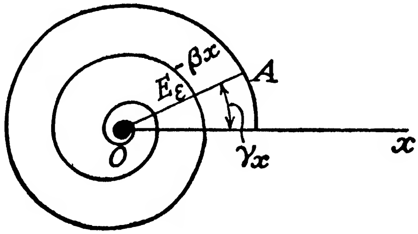

FIG. 24

Fig. 24 is drawn for a line such that $\beta = 0{\cdot}0024$, and $\gamma = 0{\cdot}0176$ radian per mile. The voltages are tabulated in Table I.

TABLE I

x	$E_T\varepsilon^{-\beta x}$	γ	x	$E_T\varepsilon^{-\beta x}$	γ
0	10	0°	250	5·64	252°
10	9·7	10°	300	4·88	303°
20	9·38	20°	350	4·43	353°
40	9·12	40°	400	3·85	404°
80	8·27	80·5°	500	3·03	505°
100	7·87	101°	600	2·38	606°
150	6·95	151°	700	1·88	707°
200	6·23	202°			

It is here noticed that at $2\pi/\gamma = 2\pi/0{\cdot}0176 = 355$ miles the voltage is in phase with the sending voltage.

It is customary to define β as the attenuation factor and γ as the wavelength factor.

EFFECT OF FREQUENCY ON β AND γ. The value of β may be written as

$$2\beta^2 = \sqrt{[(R^2 + \omega^2L^2)\,(G^2 + \omega^2C^2)]} + RG - \omega^2LC$$

On differentiating this expression with respect to ω, we obtain

$$\frac{d(2\beta^2)}{d\omega} = \omega\left[L\left(\frac{G^2 + \omega^2C^2}{R^2 + \omega^2L^2}\right)^{\frac{1}{4}} - C\left(\frac{R^2 + \omega^2L^2}{G^2 + \omega^2C^2}\right)^{\frac{1}{4}}\right]^2$$

The value of β increases with increase of frequency, and has its minimum value when the frequency is zero. If $CR = LG$, then the value of β is given by $\sqrt{(RG)}$, which is independent of frequency. Likewise it may be shown that for the wavelength factor γ when $CR = LG$, then $\gamma = \omega\sqrt{(LC)}$. The

velocity of propogation is now $1/\sqrt{(LC)}$, which is also independent of frequency.

Thus for a line where $CR = LG$ there is no distortion of attenuation or of velocity due to frequency change. However, phase distortion may still exist. As actual line constants do not meet this specification of $CR = LG$, inductive coils known as *loading coils* are placed in the line. The usual type of line is composed of two parallel copper wires. The capacity and inductance per unit length are readily calculable. For lines of diameter b and separation d, with $d \gg b$, we have in air that

$$C = \frac{\pi\kappa_0}{\text{lgn. }(2d/b)} \quad \text{and} \quad L = \frac{\mu_0}{\pi}\log_\epsilon\left(\frac{2d}{b}\right)$$

Usually R and G are negligibly small, so

$$\alpha = \pm j\gamma = j\omega(LC)^{\frac{1}{2}}$$

$$\therefore \gamma = \omega\sqrt{(\kappa_0\mu_0)} \text{ and } \omega/\gamma = 1/\sqrt{(\kappa_0\mu_0)} = 3 \times 10^8 \text{ m/sec.}$$

Also the surge impedance is

$$\sqrt{\frac{L}{C}} = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\kappa_0}} \cdot \log_\epsilon \frac{2d}{b}$$

$$= 276 \log_{10} \cdot \left(\frac{2d}{b}\right) \text{ohm}$$

For high-frequency work, coaxial cables are now used. They consist of concentric conductors separated by porcelain spacers. The large area reduces high-frequency resistance loss, and as the electric and magnetic fields are confined to the region between conductors, interaction is reduced. Here

$$C = \frac{2\pi\kappa_0}{\log_\epsilon (r_1/r_2)} \quad \text{and} \quad L = \frac{\mu_0}{2\pi}\log_\epsilon\left(\frac{r_1}{r_2}\right)$$

The velocity of propagation is as before, but now the surge impedance reduces to $138 \log_{10} (r_1/r_2)$.

3. **Lines in Series.** We may rewrite the line equations as

$$E_T = E_R A + I_R B$$
$$I_T = I_R A + E_R C$$

where $A = \cosh \alpha l$, $B = \sqrt{(Z/Y)} \cdot \sinh \alpha l$, and $C = \sqrt{(Y/Z)}$ $\sinh \alpha l$.

For two lines of different constants connected in series, where the first has a surge impedance Z_2 and the second a surge

impedance Z_1, let the voltage and current at the receiving end be E_R and I_R respectively. Then the voltage and current at the sending end of that cable will be E_{T2} and I_{T2} respectively, or

$$E_{T2} = E_R A_1 + I_R B_1$$

and
$$I_{T2} = I_R A_1 + E_R C_1$$

The voltage and current at the sending end of the first section are

$$E_{T1} = E_{T2} A_2 + I_{T2} B_2$$

and
$$I_{T1} = I_{T2} A_2 + E_{T2} C_2$$

Thus by simple substitution we have

$$E_{T1} = E_R[A_1 A_2 + C_1 B_2] + I_R[B_1 A_2 + A_1 B_2]$$

and
$$I_{T1} = I_R[A_1 A_2 + B_1 C_2] + E_R[A_2 C_1 + A_1 C_2]$$

If now the impedance of the second line is equal to its surge impedance, i.e. the line is of infinite length, then

$$E_{T2}/I_{T2} = Z_1 = 1/Y_1$$

The impedance of the whole system is

$$Z = \frac{E_{T1}}{I_{T1}} = \frac{A_2 + B_2 Y_1}{Y_1 A_2 + C_2}$$

$$= \frac{\cosh \alpha l + Z_2 Y_1 \sinh \alpha l}{Y_1 \cosh \alpha l + (1/Z_2) \sinh \alpha l}$$

$$\therefore Z = Z_2 \left[\frac{Z_1 \cosh \alpha l + Z_2 \sinh \alpha l}{Z_2 \cosh \alpha l + Z_1 \sinh \alpha l} \right]$$

When Z_2 is almost equal to Z_1, say $Z_1 = Z_2 + Z_a$, it is seen that

$$Z = Z_2 \left[\frac{\cosh \alpha l + \sinh \alpha l + (Z_a/Z_2) \cosh \alpha l}{\cosh \alpha l + \sinh \alpha l + (Z_a/Z_2) \sinh \alpha l} \right]$$

$$= Z_2 \left[1 + \frac{(Z_a/Z_2)(\cosh \alpha l - \sinh \alpha l)}{\cosh \alpha l + \sinh \alpha l + (Z_a/Z_2) \sinh \alpha l} \right]$$

Remembering that Z_a/Z_2 is small, we have

$$Z \cong Z_2 \left[1 + \frac{Z_a}{Z_2} \cdot \frac{\cosh \alpha l - \sinh \alpha l}{\cosh \alpha l + \sinh \alpha l} \right]$$

$$= Z_2 + Z_a \varepsilon^{-2\alpha l}$$

4. General Solution in Terms of Characteristic Vibrations. In order to determine the value of the voltage and current at any time and place along a line, we may use either the expansion theorem and obtain a result in characteristic vibrations or the equivalency of operators and obtain a result in waves. The first method, dealt with here, yields results which usually converge slowly, so are cumbersome to handle in calculations. With the following notation

$$v^2 = 1/LC,\ a = R/2L,\ b = G/2C,\ \rho = a + b, \text{ and } \sigma = a - b$$

the general transmission equations, with x measured from the transmitter, may be written as

$$e_x = A\varepsilon^{\alpha x} + B\varepsilon^{-\alpha x} \qquad (243)$$

$$i_x = \frac{1}{Lv}\sqrt{\left(\frac{p + 2b}{p + 2a}\right)}(B\varepsilon^{-\alpha x} - A\varepsilon^{\alpha x}) \qquad (244)$$

$$\alpha = (1/v)\sqrt{[(p + \rho)^2 - \sigma^2]} \qquad (245)$$

Open-circuited Line. In this case we have

$$e_x = E_0 \text{ at } x = 0$$
$$i = 0 \quad \text{at } x = l$$

as the boundary conditions. The voltage and current equations for an applied steady voltage become

$$e_x = E_0\frac{\cosh\alpha(l - x)}{\cosh\alpha l}\cdot(1) \qquad (246)$$

$$i_x = \frac{E_0}{Lv}\left(\frac{p + 2b}{p + 2a}\right)^{\frac{1}{2}}\cdot\frac{\sinh\alpha(l - x)}{\cosh\alpha l}\cdot(1) \qquad (247)$$

Considering the voltage equation, we have

$$Z(p) = \cosh\alpha l$$

With $Z(p) = 0$ we get that

$$\alpha l = \pm jn(\pi/2)^*$$

But $$\alpha^2v^2 = (p + \rho)^2 - \sigma^2 = -\frac{v^2n^2\pi^2}{4l^2}$$

$$\therefore p = -\rho \pm \sqrt{\left(\sigma^2 - \frac{v^2n^2\pi^2}{4l^2}\right)} = -\rho \pm j\beta_n, \text{ say} \qquad (248)$$

* n is odd.

Now
$$\frac{dZ}{dp} = \frac{dZ}{d\alpha} \cdot \frac{d\alpha}{dp} = l \sinh \alpha l \, . \, \frac{p+\rho}{av^2}$$
$$= (p+\rho)\frac{l^2}{v^2}\left[\frac{\sin (n\pi/2)}{n\pi/2}\right]$$
$$= \pm j\beta_n \cdot \frac{l^2}{v^2} \cdot \frac{\sin (n\pi/2)}{n\pi/2} \quad . \quad (249)$$

With $p = 0$ we obtain
$$\frac{Y(0)}{Z(0)} = \frac{\cosh \sqrt{\left(\frac{\rho^2-\sigma^2}{v^2}\right)} \, . \, l - x}{\cosh \sqrt{\left(\frac{\rho^2-\sigma^2}{v^2}\right)} \, . \, l}$$
$$= \frac{\cosh \{[2(l-x)\sqrt{(ab)}]/v\}}{\cosh \{[2l\sqrt{(ab)}]/v\}} \quad . \quad . \quad (250)$$

Thus the complete expression for the expansion is
$$e_x = E_0 \left[\frac{\cosh \frac{2\sqrt{(ab)}}{v} \, . \, (l-x)}{\cosh \frac{2\sqrt{(ab)}}{v} \, . \, l} + \sum_{n=1}^{n=\infty} \frac{\cos \frac{n\pi}{2l}(l-x) \, . \, \varepsilon^{(-\rho \pm j\beta_n)t}}{\pm j\beta_n(-\rho \pm j\beta_n)\frac{l^2}{v^2}\frac{\sin (n\pi/2)}{n\pi/2}} \right] \quad . \quad (251)$$

The summation sign may be simplified, since
$$\frac{\varepsilon^{-\rho t}}{j\beta_n}\left(\frac{\varepsilon^{j\beta_n t}}{-\rho + j\beta_n} - \frac{\varepsilon^{-j\beta_n t}}{-\rho - j\beta_n}\right) = \frac{-2\varepsilon^{-\rho t}(\rho \sin \beta_n t + \beta_n \cos \beta_n t)}{\beta_n(\rho^2 + \beta_n^{\,2})}$$
$$= \frac{-2 \cos (\beta_n t - \phi_n)}{\beta_n \sqrt{(\rho^2 + \beta_n^{\,2})}} \varepsilon^{-\rho t}$$
where $\tan \phi_n = \rho/\beta_n$.

The expression for the voltage becomes
$$e_x = E_0 \left\{ \frac{\cosh [2\sqrt{(ab)}\,(l-x)/v]}{\cosh [2(\sqrt{ab})l/v]} - \frac{\pi v^2}{l^2}\varepsilon^{-\rho t}\sum_{n=1}^{n=\infty}\frac{n \sin (n\pi x/2l) \cos (\beta_n t - \phi_n)}{\beta_n\sqrt{(\rho^2 + \beta_n^{\,2})}} \right\} \quad . \quad (252)$$

At $x = l$ we obtain the receiver volts as

$$E_R = E_0 \left\{ \frac{1}{\cosh [2(\sqrt{ab})l/v]} - \frac{\pi v^2}{l^2} \varepsilon^{-\rho t} \sum_{n=1}^{n=\infty} \frac{n \cos (\beta_n t - \phi_n)}{\beta_n \sqrt{(\rho^2 + \beta_n^2)}} \right\} \quad (253)$$

The value of the current may be obtained from

$$- di/dx = (G + Cp)e$$

This gives

$$- i_x = E_0 \left\{ \frac{Gv}{2\sqrt{ab}} \frac{\sinh [2(\sqrt{ab})(l - x)/v]}{\cosh [2(\sqrt{ab})l/v]} + \frac{2v^2}{l} \varepsilon^{-\rho t} \right.$$

$$\left. \sum_{n=1}^{n=\infty} \frac{\cos (n\pi x/2l)}{\beta_n} \left[\frac{G \cos (\beta_n t - \phi_x)}{\sqrt{(\rho^2 + \beta_n^2)}} - C \sin \beta_n t \right] \right\} \quad (254)$$

With $G = L = 0$ the cable is the Heaviside "ideal." On making these substitutions, i.e. $\alpha = \sqrt{(RCp)}$, we get

$$E_x = E_0 \left[1 + \frac{2}{\pi} \sum_{n=1}^{n=\infty} \frac{\cos (n\pi x/l)}{(-1)^{(n-1)}2 \,.\, n} \varepsilon^{-n^2\pi^2 t/4CRl^2} \right] \quad . \quad (255)$$

$$I_x = - E_0 \sum_{n=1}^{n=\infty} (-1)^{\frac{n-1}{2}} \cdot \frac{2}{Rl} \sin \frac{n\pi x}{l} \,.\, \varepsilon^{-n^2\pi^2 t/4CRl^2} \,. \quad (256)$$

$$= \frac{2E_0}{Rl} [\varepsilon^{-\pi^2 t/4CRl^2} + \varepsilon^{-9\pi^2 t/4CRl^2} + \ldots]$$

For very small values of t we get

$$i = (2E_0/Rl)\, \varepsilon^{-\pi^2 t/4CRl^2}$$

Characteristic Frequency of Line. The second term of equations (252) and (254) contains a component having an angular velocity $\beta_n t$. The frequency will be

$$f_n = \frac{\beta_n}{2\pi} = \frac{1}{2\pi} \sqrt{\left(\frac{v^2 n^2 \pi^2}{l^2} - \sigma^2 \right)} \quad . \quad . \quad (257)$$

Consider the case when $\sigma = 0$, i.e. $R/L = G/C$. Then we have

$f_n = vn/2l$, $f_1 = v/2l = 1/2l\sqrt{(LC)}$, $f_2 = 3/2l\sqrt{(LC)}$, etc.

This gives the frequency characteristic of the line.

EARTHED LINE. We give one more example of this method, that of an earthed line. The boundary conditions are

$$E = E_T \text{ at } x = 0$$
$$E = 0 \quad \text{at } x = l$$

So the equations are

$$E_x = E_T \frac{\sinh \alpha\,(l-x)}{\sinh \alpha l} \cdot (1) \qquad . \qquad . \quad (258)$$

$$I_x = \frac{E_T}{Lv}\sqrt{\left(\frac{p+2b}{p+2a}\right)}\frac{\cosh \alpha(l-x)}{\sinh \alpha l} \cdot (1) \qquad . \quad (259)$$

As the development is similar to that of Section 3, we merely sketch the method.

$$Z(p) = 0 \text{ gives } \alpha l = jn\pi \quad (n = 1, 2, 3, \ldots)$$

$$p = -\rho \pm \sqrt{(\sigma^2 - n^2v^2\pi^2/l^2)} = -\rho \pm j\beta_n$$

$$Z'(p) = \frac{l^2}{v^2}(p+\rho)\frac{\cosh \alpha l}{\alpha l} = \pm j\beta_n \frac{l^2}{v^2}\frac{\cos (n\pi/2)}{n\pi/2}$$

$$\frac{Y(0)}{Z(0)} = \frac{\sinh [2(\sqrt{ab})\,(l-x)/v]}{\sinh [2(\sqrt{ab})l/v]} \qquad . \qquad . \qquad . \quad (260)$$

Expansion gives

$$E_x = E_T\Bigg[\frac{\sinh [2(\sqrt{ab})\,(l-x)/v]}{\sinh [2(\sqrt{ab})l/v]} - \frac{2v^2\pi}{l^2}\varepsilon^{-\rho t}$$
$$\sum \frac{n}{\beta_n}\sin\left(\frac{n\pi x}{l}\right)\frac{\cos (\beta_n t - \phi_n)}{\sqrt{(\rho^2 + \beta_n{}^2)}}\Bigg] . \quad (261)$$

$$I_x = E_T\Bigg\{\frac{G\cosh [2(\sqrt{ab})\,(l-x)/v]}{\sinh [2(\sqrt{ab})l/v]} - \frac{2v^2}{l}\varepsilon^{-\rho t}$$
$$\sum \frac{\cos (n\pi x/l)}{\beta_n}\left[\frac{G\cos (\beta_n t - \phi_n)}{\sqrt{(\rho^2 + \beta_n{}^2)}} - C\sin \beta_n t\right]\Bigg\} . \quad (262)$$

The characteristic frequencies now are given by

$$f_n = nv\pi/2\pi l \text{ for } R/L = G/C$$

$$\therefore f_1 = 1/2l\sqrt{(LC)},\ f_2 = 1/\sqrt{(LC)}, \text{ etc.} \qquad . \quad (263)$$

5. **Difference Equations: Recurrent or Inactive Networks.** Certain networks used in practice are built up, and so their characteristics are lumped or non-uniform. Such networks

must be considered as discrete systems (a smooth system is the limit of a discrete system). To handle such systems we resort to *difference equations*. Examples are now given of this method.

ARTIFICIAL LINE. To simulate actual line conditions an artificial line is built up, so permitting laboratory investigation. We will consider the line to be built in T-section. For the pillar we will use $1/(G+j\omega C) = B$ and on the top we will use equal impedances $(R + j\omega L) = 2A$. The general arrangement is as shown in Fig. 25. For such a network to simulate the smooth

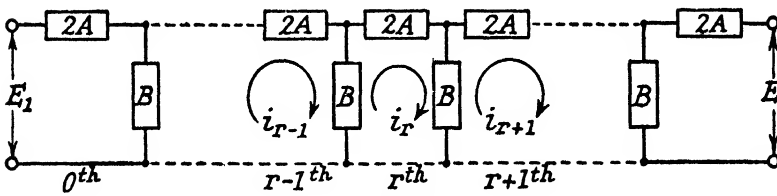

FIG. 25

line it is essential that the number of sections should be large. But when terminal conditions only are required, then it is possible to accomplish this in one section, provided that the following values are adopted—

$$2A = Z_0\left(\frac{\cosh \alpha l - 1}{\sinh \alpha l}\right) = Z_0 \tanh (\alpha l/2)$$

$$B = Z_0/\sinh \alpha l$$

where Z_0 and α are determined from smooth line characteristics.

Let the network have n sections. From Kirchhoff's laws, for the r^{th} section we have

$$i_r 2A + i_r 2B - i_{r-1}B - i_{r+1}B = 0 \quad . \quad . \quad (264)$$

or
$$i_r[2A + 2B] - B[i_{r-1} + i_{r+1}] = 0 \quad . \quad . \quad (265)$$

Let
$$i_r = P\varepsilon^{\alpha r} + Q\varepsilon^{-\alpha r} \quad . \quad . \quad (266)$$

Then by substitution we get

$$[(2A + 2B) - B(\varepsilon^{\alpha} + \varepsilon^{-\alpha})]\ (P\varepsilon^{\alpha r} + Q\varepsilon^{-\alpha r}) = 0 \quad (267)$$

$$\therefore\ 2(A + B)/B = \varepsilon^{\alpha} + \varepsilon^{-\alpha} = 2 \cosh \alpha \quad . \quad (268)$$

i.e. $1 + A/B = \cosh \alpha$ is the value of α to satisfy equation (266).

For the determination of P and Q we proceed as follows. In

the initial section the voltage is E_1 and $r = 0$; for the nth section the voltage is E_2 and $r = n$. Then

$$E_1 = P(A + B - B\varepsilon^{\alpha}) + Q(A + B - B\varepsilon^{-\alpha}) \quad . \quad . \quad (269)$$

$$E_2 = P(A + B - B\varepsilon^{-\alpha})\varepsilon^{n\alpha} + Q(A + B - B\varepsilon^{\alpha})\varepsilon^{-n\alpha} \quad (270)$$

Whence

$$P = \frac{E_2 + E_1\varepsilon^{-n\alpha}}{2B \sinh \alpha \sinh n\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (271)$$

and $$Q = \frac{E_2 + E_1\varepsilon^{n\alpha}}{2B \sinh \alpha \sinh n\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (272)$$

$$\therefore I_R = \left[\frac{E_2 + E_1\varepsilon^{-n\alpha}}{2B \sinh \alpha \sinh n\alpha}\right]\varepsilon^{r\alpha} + \left[\frac{E_2 + E_1\varepsilon^{-n\alpha}}{2B \sinh \alpha \sinh n\alpha}\right]\varepsilon^{-r\alpha}$$

$$= \frac{E_2 \cosh r\alpha + E_1 \cosh (n - r)\alpha}{B \sinh n\alpha \sinh \alpha} \quad . \quad . \quad . \quad (273)$$

When n is very large, this equation reduces to

$$I_R = \frac{E_1}{B} \frac{\cosh (n - r)\alpha}{\sinh n\alpha \sinh \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (274)$$

Application of an Impulse. We now consider the application of an impulse voltage E at the transmitter. Then

$$I = \frac{E}{B} \frac{\cosh (n - r)\alpha}{\sinh n\alpha \sinh \alpha} \cdot (1) \quad . \quad . \quad (275)$$

In subsequent work the leakage conductance will be neglected so that

$$2A = Lp + R; \; B = 1/Cp$$

$$\therefore \cosh \alpha = 1 + Cp(Lp + R)/2 \quad . \quad . \quad (276)$$

Using the method of Section 3, Chapter VII, we proceed as follows—

$$Z(p) = \sinh n\alpha \sinh \alpha \quad . \quad . \quad (277)$$

For $Z(p) = 0$ we obtain $\alpha = js\pi/n$, where $s = 1, 2, 3 \ldots$ Substituting in quotation (276), we have

$$\cos (s\pi/n) = 1 + Cp(Lp + R)/2 \quad . \quad . \quad . \quad . \quad . \quad (278)$$

$$\therefore p = -\frac{R}{2L} \pm \sqrt{\left[\frac{R^2C^2 - 8LC(1 - \cos 3\pi/n)}{2LC}\right]} \quad . \quad (279)$$

$$= -\rho \pm j\beta_s$$

where $\rho = R/2L$ and $\beta_s = \sqrt{[4v^2 \sin^2 (s\pi/2n) - \rho^2]}$.

Now, $$Z'(p) = [n \cosh n\alpha \sinh \alpha + \cosh \alpha \sinh n\alpha](d\alpha/dp)$$
$$= n \cosh n\alpha \sinh \alpha \, . \, (d\alpha/dp) \quad . \quad . \quad . \quad (280)$$

Also from (276) we have

$$\sinh \alpha \, . \, (d\alpha/dp) = LCp + RC/2$$

$$\therefore Z'(p) = n(LCp + RC/2) \cosh n\alpha$$
$$= nLC(p + \rho) \cos s\pi = (-1)^s nLC(p + \rho) \quad . \quad (281)$$

When p approaches 0, i.e. when α approaches 0—

$$\sinh \alpha \sinh n\alpha \rightarrow n\alpha^2$$

Now, $$\cosh \alpha = 1 + \tfrac{1}{2}\alpha^2 \ldots = 1 + \tfrac{1}{2}Cp(Lp + R)$$

$$\therefore \alpha^2 = Cp(Lp + R) \quad . \quad . \quad (282)$$

$$\frac{Y(0)}{Z(0)} = \frac{Cp}{nCp(Lp + R)} = \frac{1}{nR} \quad . \quad . \quad (283)$$

So we have

$$i_R = E\left[\frac{1}{nR} + \sum \frac{Cp_n \cosh (n - r)\alpha \, . \, \varepsilon^{p_n t}}{nLCp_n(p_n + \rho) \cos s\pi}\right]$$

$$= E\left[\frac{1}{nR} + \frac{\varepsilon^{-\rho t}}{nL} \sum_{s=1}^{s=n} \frac{\cosh \overline{n - r}\alpha \, . \, \varepsilon^{\pm j\beta_s t}}{(p_n + \rho)(-1)^s}\right] \quad . \quad (284)$$

For the nth section this reduces to

$$i_n = E\left[\frac{1}{nR} + \frac{\varepsilon^{-\rho t}}{nL} \sum_{s=1}^{s=n} \frac{\varepsilon^{j\beta_s t} - \varepsilon^{-j\beta_s t}}{j\beta_s (-1)^s}\right]$$

$$= E\left[\frac{1}{nR} + \frac{2\varepsilon^{-\rho t}}{nL} \sum_{s=1}^{s=n} \frac{\sin \beta_s t}{\beta_s} (-1)^s\right] \quad . \quad (285)$$

As n is very large, we may write $\sin (s\pi/2n) = s\pi/2n$.

Then $$\beta_s = \sqrt{\left(\frac{v^2\pi^2 s^2}{n^2} - \rho^2\right)} \quad . \quad . \quad . \quad (286)$$

This equation is of the same form as that obtained for a smooth line. (See equation (262).)

6. **Filter Circuit.** In communication work it is necessary at times to accept only one frequency band and to reject all others. The network which accomplishes this purpose is known

as a *filter circuit.* The ideal filter circuit would allow all signals in a desired frequency band to pass without attenuation and would attenuate all others to zero.

Several circuits may be used as filter circuits, but as their actions are, within limits, the same, we select one of the form shown in Fig. 26. It will be seen that it is of the same form as

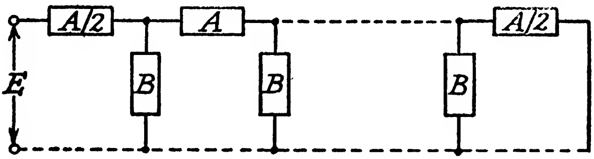

FIG. 26

the artificial line except that here the circuit is properly terminated.

Let there be n sections in the network. The investigation proceeds along lines the same as those in Section 5, and we obtain for the rth section that

$$i_r = P\varepsilon^{r\alpha} + Q\varepsilon^{-r\alpha} \quad . \quad . \quad (287)$$

where $2 \cosh \alpha = (A/B + 2)$.

From terminal conditions we have

$$E = \tfrac{1}{2}i_0 A + B(i_0 - i_1) \text{ for the first section} \quad . \quad (288\text{A})$$

and

$$0 = \tfrac{1}{2}i_n A + B(i_{n-1} - i_n) \text{ for the last section} \quad (288\text{B})$$

By substitution of (287), we have that

$$i_R = \frac{E}{B} \cdot \frac{\cosh (n - r)\alpha}{\sinh n\alpha \sinh \alpha} \quad . \quad . \quad . \quad . \quad (289)$$

Actually we are only interested in the nth section, where

$$i_n = \frac{E}{B} \cdot \frac{1}{\sinh n\alpha \sinh \alpha} \cdot (1) \quad . \quad . \quad . \quad (290)$$

LOW-PASS CIRCUIT. For a *low-pass* filter circuit we have $A = R + pL$ and $B = 1/Cp$, so that

$$2 \cosh \alpha = (R + pL)Cp + 2$$

For a steady voltage E . (1), applied at the sending end, the current at the receiving end is given by

$$i_n = E\left[\frac{1}{nR} + \frac{2\varepsilon^{-\rho t}}{nL}\sum_{s=1}^{s=n}\frac{\sin \beta_s t}{(-1)^s \beta s}\right] \quad . \quad . \quad (291)$$

If now an a.c. voltage such as $E\varepsilon^{j\omega t}$. (*1*) is applied, we then have

$$i_n = E\left[\frac{\varepsilon^{j\omega t}}{Z(j\omega)} + \sum_{s=1}^{s=n} \frac{Cp_s}{(p_s^2 + \omega^2)} \frac{\varepsilon^{p_s t}}{Z'(p)_{ps}}\right]$$

The second term becomes

$$\frac{E\varepsilon^{-\rho t}}{nL} \sum_{s=1}^{s=n} \frac{(-\rho \pm j\beta_s)\,\varepsilon^{\pm j\beta_s t}}{\pm j\beta_s[(-\rho \pm j\beta_s) + \omega^2] \cos s\pi} \quad . \quad . \quad (292)$$

or

$$\frac{2E\varepsilon^{-\rho t}}{nL} \sum_{s=1}^{s=n} \frac{[(\rho^2 + \beta_s^2)^2 + \omega^2(\rho^2 - \beta_s^2)] \sin \beta_s t - 2\rho\beta_s \cos \beta_s t}{\beta_s[(\rho^2 + \beta_s^2)^2 + 2\omega^2(\rho^2 - \beta_s^2) + \omega^4] \cos s\pi} \quad (293)$$

It is seen that the last term of the expression gives the transient components, the number of which is identical with the number of meshes. The attenuation is given as $\rho = R/2L$ and the responsive frequency as

$$f_s = \beta_s/2\pi$$
$$= (1/2\pi)\sqrt{[4v^2 \sin^2 (s\pi/2n) - \rho^2]}$$

Neglecting ρ^2, we have

$$f_s = (v/\pi) \sin (s\pi/2n)$$

or

$$= \frac{\sin (s\pi/2n)}{\pi\sqrt{(LC)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (294)$$

At $s = n$—

$$f_n = 1/\pi\sqrt{(LC)} = \text{the upper frequency}$$

When $s = 1$—

$$f_1 = \frac{\sin (\pi/2n)}{\pi\sqrt{(LC)}} = \text{the lower frequency}$$

i.e. if we increase n we lower the frequency f_1, which will pass through. The value of f_n, not being affected by the number of sections, will remain the same.

Other filter circuits may be considered along these lines.

7. **Potential Distribution along an Insulator String.** In a suspension-type insulator let Z_1 and Z_2 represent the impedances between tie pins and between pin and earth respectively.

Then we have for n disks the equivalent circuit of Fig. 27. As in Section 5, we obtain

$$(Z_1/Z_2) + 2 = 2 \cosh \alpha$$

$$e_r = P\varepsilon^{r\alpha} + Q\varepsilon^{-r\alpha} \quad . \quad . \quad . \quad (295)$$

$$e_r/Z_2 = [(e_{r-1} - e_r) - (e_r - e_{r+1})]/Z_1$$

For $r = 0$ we have $\quad P + Q \quad = 0$

For $r = n$ we have $\quad P\varepsilon^{n\alpha} + Q\varepsilon^{-n\alpha} = E$

$$\therefore E = 2P \sinh n\alpha$$

and

$$e_r = E\frac{\sinh r\alpha}{\sinh n\alpha} \quad . \quad . \quad . \quad . \quad (296)$$

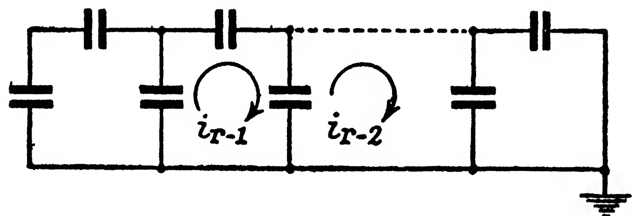

FIG. 27

The voltage across the rth disk is

$$V_R = e_r - e_{r-1}$$

or

$$V_r = \frac{2E}{\sinh n\alpha}[\sinh \tfrac{1}{2}\alpha \cosh \alpha(r - \tfrac{1}{2})]$$

$$\frac{V_r}{V_{r+1}} = \frac{\cosh (r - \frac{1}{2})\alpha}{\cosh (r + \frac{1}{2})\alpha} \quad . \quad . \quad . \quad . \quad (297)$$

If now Z_1 and Z_2 are represented by pure capacities, then

$$2 \cosh \alpha = C_2/C_1 + 2$$

i.e. a real number.

Under these conditions the voltage would taper off exponentially from the line side to earth. In practice it is found that this is not so; the voltage curve rises on the earth side. This is readily explicable when it is realized that Z_1 should include surface leakage due to dirt, moisture, and local surface ionization. Extant literature gives no definite formula to express all these quantities simply, so we take

$$Z_1 = \frac{R}{1 + j\omega C_1 R}$$

where R is a resistance shunting the condenser between pins.

Now $$\cosh \alpha = 1 + \frac{C_2 R}{1 + j\omega C_1 R}$$

so that α is of the form $A + jB$ and the distribution is no longer exponential but may have a definite minimum. This is the distribution which holds in practice.

8. **Distribution of Voltage along a Transformer Layer Winding.** In a transformer winding, non-uniformity of voltage distribution has long been recognized, and has resulted in the

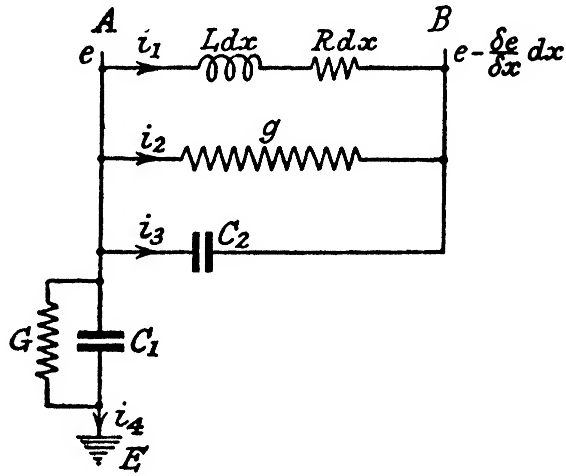

FIG. 28

use of reinforced insulation for the end coils and of a solid ring for the end coil. Recent work, however, has thrown some doubt on the efficiency of both these methods. Due to the complexity of the problem, we make simplifying assumptions to bring our discussion within reasonable limits.

In any single-layer transformer winding the equivalent circuit could be represented as in Fig. 28, where we have the following quantities—

(i) Resistance of winding. If r is the resistance per unit length of the winding, we have an elemental resistance of $r \,.\, dx$.

(ii) Inductance of winding, due to ϕ (the total flux in the core, i.e. the useful flux plus the leakage flux, the latter varying along the winding).

(iii) Capacity of winding to core. As the core is usually earthed, this will be the capacity to earth. If C is the total

winding capacity to earth, then the elemental capacity will be $(C/l)dx = C_1 \, . \, dx$.

(iv) Capacity between turns. This will be a series arrangement, and so, if C is the total self-capacity, the elemental capacity will be $Cl/dx = C_2/dx$.

(v) Conductances existing between coils, and between coils and earth. These we represent as g and G respectively.

EQUATIONS. Let e and $\left(e - \frac{\partial e}{\partial x} \, . \, dx\right)$ be the voltages at A and B respectively. Then we have

$$i_4 = (G + C_1 p)e \quad . \quad . \quad . \quad . \quad (298)$$

$$i_3 = \frac{C_2}{dx} \, . \, p \, . \, de \quad . \quad . \quad . \quad . \quad (299)$$

$$i_2 = g \frac{de}{dx} \quad . \quad . \quad . \quad . \quad . \quad (300)$$

$$\frac{de}{dx} = Ri_1 + \frac{n}{10^8} \frac{d\phi}{dt} \quad . \quad . \quad . \quad (301)$$

$$i_4 = \frac{d}{dx}(i_1 + i_2 + i_3) \quad . \quad . \quad . \quad (302)$$

where n is the number of turns per unit length of winding.

In equation (301) the value of ϕ may be split up into

(i) Mutual flux (ϕ_m) linking the entire winding and invariable with respect to x.

(ii) A flux (ϕ_e) linking part of the windings and variable with respect to x.

(iii) In a multi-layer winding, the variation of the flux linkages across a section must be taken into account. For a single-layer winding*

$$\begin{aligned} \phi &= \phi_m + \phi_e \\ &= \phi_m + \frac{0{\cdot}4\pi(l.m.t.)n}{W} \int_0^x \int_x^l i_1 dx_1 dx_2 \end{aligned}$$

where W = equivalent air-gap length,

$l.m.t.$ = length of mean turn.

* BEWLEY: *Trans. A.I.E.E.* (1932), Vol. 51, p. 299.

On differentiating equation (301), we get*

$$\frac{\partial^4 e}{\partial x^4} = R\frac{\partial^3 i_1}{\partial x^3} - \frac{0{\cdot}4\pi(l.m.t.)n^2}{W \times 10^8}\frac{\partial^2 i_1}{dx\,.\,dt}$$

$$= R\frac{\partial^3 i_1}{\partial x^3} - \frac{L}{l^3}\frac{\partial^2 i_1}{\partial x\,.\,\partial t} \quad . \quad . \quad . \quad . \quad (303)$$

where $L = \dfrac{0{\cdot}4\pi n^2(l.m.t)l^3}{W \times 10^8}$.

Substituting equations (298), (299), and (300) in equation (302), we have

$$\begin{aligned}\Delta i_1 &= i_4 - \Delta(i_2 + i_3)\\ &= (G + C_1 p)e - \Delta(g\Delta e + C_2 p\Delta e)\\ &= (G + C_1 p)e - g\Delta^2 e - C_2 p\Delta^2 e \text{ where } \Delta = \frac{d}{dx} \qquad (304)\end{aligned}$$

So we have equation (303) in the following form—

$$\begin{aligned}\Delta^4 e = R[(G + C_1 p)\Delta^2 e - (g + C_2 p)\Delta^4 e]&\\ - \frac{L}{l^3}p[(G + C_1 p)e - (g + C_2 p)\Delta^2 e]&\end{aligned}$$

On collecting terms we have

$$\begin{aligned}\Delta^4[1 + R(g + C_2 p)e - \Delta^2[R(G + C_1 p) - \frac{L}{l^3}p(g + C_2 p)]&\\ + \frac{L}{l^3}p(G + C_1 p)e = 0 \quad . \quad (305)&\end{aligned}$$

From the general equation (305) we are able to deduce the voltage distribution in the winding.

INITIAL VOLTAGE DISTRIBUTION. Initially the voltage distribution will depend solely on the series capacity, so that we have

$$i_4 = \Delta i_3 \quad . \quad . \quad . \quad . \quad (306)$$

$$i_3 = C_2 p\Delta e \quad . \quad . \quad . \quad (307)$$

$$i_4 = C_1 pe \quad . \quad . \quad . \quad . \quad (308)$$

Hence $\qquad C_2 p\Delta^2 e = C_1 pe$

or $\qquad \Delta^2 e = \dfrac{C_1}{C_2}e = m^2 e \quad . \quad . \quad . \quad (309)$

* BLUME AND BOYAJIAN: *Trans. A.I.E.E.* (1919), Vol. 38, p. 577.

The solution of this equation is given by

$$e_x = A\varepsilon^{mx} + B\varepsilon^{-mx} \quad . \quad . \quad . \quad (310)$$

$$i_3 = C_2 pm[A\varepsilon^{mx} - B\varepsilon^{-mx}] \quad . \quad . \quad (311)$$

where A and B are constants dependent on terminations. If the neutral is earthed through an impedance $Z(p)$, we get at $x = 0$ $\qquad e = E$

and at $x = l$ $\qquad e = i_3 \,.\, Z(p)$

So we have that $\qquad E = A + B$

and $i_3 Z(p) = Z(p) \,.\, pC_2 m[A\varepsilon^{ml} B\varepsilon^{-ml}] = A\varepsilon^{ml} + B\varepsilon^{-ml}$

whence

$$A = \frac{1}{2E}[Z(p) \,.\, \sqrt{(C_1C_2)} \,.\, p + 1]\varepsilon^{-ml} \times [Z(p)\sqrt{(C_1C_2)} \,.\, p \cosh ml - \sinh ml]^{-1}$$

$$B = \frac{1}{2E}[Z(p)\sqrt{(C_1C_2)} \,.\, p + 1]\varepsilon^{ml} \times [Z(p)\sqrt{(C_1C_2)}p \cosh ml - \sinh ml]^{-1}$$

Hence

$$e_x = E \,.\, \frac{Z(p)\sqrt{(C_1C_2)} \,.\, p \cosh m(l-x) - \sinh m(l-x)}{Z(p)\sqrt{C_1C_2}) \,.\, p \cosh ml - \sinh ml} \quad (312)$$

when $Z(p) = 0$, we get the voltage distribution, with the neutral dead earthed, as

$$e_x = E \,.\, \frac{\sinh m(l-x)}{\sinh ml} \cong \frac{l-x}{l} \,.\, E \quad . \quad . \quad (313)$$

When $Z(p) = \infty$, we get the voltage distribution, when the neutral is isolated, as

$$e_x = E\,\frac{\cosh m(l-x)}{\cosh ml} \cong E \quad . \quad . \quad . \quad (314)$$

It is seen that from equations (313) and (314) that the voltage distribution along the winding will be exponential in form, ranging from 0 to E in the case of the earthed neutral, and from E to E in the case of the isolated neutral.

Final Distribution. The final distribution of voltage will depend on the resistance and conductance elements of the winding. Now we have

$$i_4 = Ge = \Delta(i_1 + i_2) = \Delta\left(\frac{1}{R}\Delta e + g\Delta e\right)$$

$$\therefore\ \Delta^2 e = \frac{RG}{1 + gR}\,e = n^2 e \quad . \quad . \quad . \quad . \quad (315)$$

The solution of the equaton is—

$$e = A\varepsilon^{nx} + B\varepsilon^{-nx}$$

$$i = \frac{n}{R}[A\varepsilon^{nx} - B\varepsilon^{-nx}] \qquad . \quad . \quad . \quad . \quad (316)$$

If the transformer is earthed through an impedance $Z(p)$, we get

$$E_x = E\left[\frac{Z(p)\frac{n}{R}\cosh n(l-x) - \sinh n(l-x)}{Z(p)\frac{n}{R}\cosh nl - \sinh nl}\right] \qquad . \quad (317)$$

So for a grounded neutral we have

$$E_x = E\frac{\sinh n(l-x)}{\sinh nl} \qquad . \quad . \quad . \quad (318)$$

As n^2 is small, we may approximate: thus

$$E_x = E[1 - (x/l)]$$

For the isolated neutral we get

$$E_x = E\frac{\cosh n(l-x)}{\cosh nl} \cong E$$

Thus the final state of voltage distribution is a straight line sloping either from E to 0 or E to E, depending on the impedance in the neutral connection.

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CHAPTER VIII

BUILDING-UP OF VOLTAGE AND CURRENT IN A LINE: REFLECTION PROBLEMS

General. A mental picture of the building-up of voltage in a line may be obtained from Section 4 of this chapter.* In the previous chapter we have seen that the expansion theorem gives a Fourier's series which converges very slowly and is useless for computation. We now investigate the problem using operational equivalences, i.e. Fourier's integrals.

1. **Infinite Line.** In Vol. II, p. 382, Heaviside gave an interesting development for the case of the infinite line. From Section 4, Chapter VII, we have seen that the current for a line earthed at the far end, when $G = 0$ is given by

$$I_\alpha = E_r\left[\frac{2v^2}{l}C\varepsilon^{-\rho t}\sum_{n=1}^{n=\infty}\cos\left(\frac{n\pi x}{l}\right)\frac{\sin\beta_n t}{\beta_n}\right] \quad . \quad . \quad (319)$$

The step in this summation is $n\pi/l = m$, say. In the limit we have $\pi/l = dm$. The equation now becomes

$$I_\alpha = E_r C\left[\frac{2v^2}{\pi}\varepsilon^{-\rho t}\int_0^\infty \cos mx\,\frac{\sin\beta_n t}{\beta_n}\,dm\right] \quad . \quad . \quad (320)$$

where
$$\beta_n = \sqrt{(v^2m^2 - \rho^2)} \quad . \quad . \quad (321)$$

To evaluate the integral, we note that

$$\frac{d}{d(\rho^2)}(\cos\beta_n t) = \frac{d}{d\beta_n}\cos\beta_n t \times \frac{d\beta_n}{d(\rho^2)}$$

$$= t\sin\beta_n t \times \tfrac{1}{2}\cdot\frac{1}{\beta_n} = \tfrac{1}{2}t\frac{\sin\beta_n t}{\beta_n} \quad . \quad . \quad (322)$$

But
$$\frac{d}{dt}\frac{\sin\beta_n t}{\beta_n} = \cos\beta_n t \quad . \quad . \quad . \quad . \quad . \quad . \quad (323)$$

So we have from (322) and (323) that

$$\frac{d}{d(\rho^2)}\cos\beta_n t = \tfrac{1}{2}t\frac{\sin\beta_n t}{\beta_n} = \tfrac{1}{2}t\left(\frac{1}{p}\cos\beta_n t\right)$$

and
$$\frac{d}{d\rho^2}\left(\frac{\sin\beta_n t}{\beta_n}\right) = \frac{d}{d\rho^2}\frac{1}{p}\cos\beta_n t = \frac{1}{p}\left(\frac{t}{2}\frac{\sin\beta_n t}{\beta_n}\right) \quad . \quad (324)$$

* Also, see HEAVISIDE: *Electro-magnetic Theory*, Vol. II, pp. 67–77.

Expansion by Maclaurin's theorem gives a result of the form

$$f(\rho^2) = f(0) + \rho^2 f(0) + \frac{\rho^4}{\underline{|2}} f''(0) + \ldots$$

where $f'(0), f''(0) \ldots$ denote differentiations with respect to ρ^2.

Here
$$f(\rho^2) = \frac{\sin \beta_n t}{\beta_n}$$

and so
$$f(0) = \frac{\sin mvt}{mv}.$$

and by (324) we have

$$f'(0) = \frac{1}{p}\frac{t}{2}\frac{\sin mvt}{mv}, \text{ etc.,}$$

or
$$\frac{\sin \beta_n t}{\beta_n} = \left[1 + \rho^2 \frac{1}{p}\frac{t}{2} + \frac{\rho^4}{2!}\left(\frac{1}{p}\frac{t}{2}\right)^2 + \ldots\right]\frac{\sin mvt}{mv}$$

or written in full—

$$\frac{\sin \beta_n t}{\beta_n} = \frac{\sin mvt}{mv} + \tfrac{1}{2}\rho^2 \int_0^t \frac{t \sin mvt}{mv} dt + \frac{(\frac{1}{2}\rho^2)^2}{2!} \int_0^t t dt \int_0^t \frac{t \sin mvt}{mv} dt + \ldots$$

So the integral has the form

$$\int_0^\infty \cos mx \frac{\sin mvt}{mv} dm = \tfrac{1}{2}\int_0^\infty \left[\frac{\sin m(vt + x) + \sin m(vt - x)}{mv}\right] dm$$

By tables—

$$\int_0^\infty \frac{\sin mx}{m} dm = \frac{\pi}{2}$$

So that the integral given above will have the value

$$\frac{\pi}{2v}[\tfrac{1}{2} + \tfrac{1}{2}] \text{ if } vt > x$$

and
$$\frac{\pi}{2v}[\tfrac{1}{2} - \tfrac{1}{2}] \text{ if } vt < x$$

To avoid the zero or singularity we will start our integration at a time $t = x/v$. We revert to the physical aspect of this step later. Now the integral becomes

$$\int_0^\infty \cos mx \frac{\sin \beta_n t}{\beta_n} dm$$
$$= \frac{\pi}{2v}\left[1 + \tfrac{1}{2}\rho^2 \int_{x/v}^t t dt + \frac{(\frac{1}{2}\rho^2)^2}{2!}\int_{x/v}^t t dt \int_{x/v}^t t dt \ldots\right]$$

Term by term, the integral becomes $\frac{1}{2}\left[t^2 - \frac{x^2}{v^2}\right]$, $\frac{1}{8}\left[t^2 - \frac{x^2}{v^2}\right]^2$, etc. Thus a series, known as a *Bessel series*, emerges from this integral, or

$$\int_0^\infty \cos mx \frac{\sin \beta_n t}{\beta_n} dm = \frac{\pi}{2v} I_0\left[\rho\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]_{(t > x/v)}$$

So that the current in the line becomes

$$i = E_T C \left\{\frac{2v^2}{\pi}\varepsilon^{-\rho t}\frac{\pi}{2v} I_0\left[\rho\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]\right\}_{(t > x/v)}$$

$$= E_T\sqrt{\left(\frac{C}{L}\right)}\varepsilon^{-\rho t} I_0[. \ . \ .]. \qquad (325)$$

Physically, no current can exist in a line until the impulse has had time to arrive, or $t = x/v$. Once the impulse arrives, the current will build up according to the above expression. For example, at $x = 0$ the current commences to build up from the instant of switch closure as

$$E_T\sqrt{(C/L)}\varepsilon^{-\rho t} I_0[\rho t]$$

but at a distance x along the line the current is zero until a time $x/v = \theta$, say, has elapsed. Thereafter it will grow as

$$E_T\sqrt{(C/L)}\varepsilon^{-\rho t} I_0[\rho(t^2 - \theta^2)^{\frac{1}{2}}]$$

2. **Operational Development.** This result could have been developed by direct operational methods as follows—

For an infinite line
$$i = \frac{V}{Lv}\sqrt{\left(\frac{p + 2b}{p + 2a}\right)}\varepsilon^{-\alpha x} \qquad (326)$$

or at the transmitting end
$$i_0 = \frac{V}{Lv}\sqrt{\left(\frac{p + 2b}{p + 2a}\right)}(1) \qquad (327)$$

for a steady applied voltage.

But
$$\frac{V}{Lv}\frac{(p + 2b)}{\sqrt{[(p + 2a)(p + 2b)]}} \cdot (1) = \frac{V}{Lv}\frac{p\left(1 + \frac{G}{Cp}\right)}{\sqrt{(p + \rho)^2 - \sigma^2]}} \cdot (1)$$

Now using the shifting theorem we have

$$\frac{p}{\sqrt{[(p + \rho)^2 - \sigma^2}} = \varepsilon^{\rho t} . \varepsilon^{-\rho t}\frac{p}{\sqrt{[(p + \rho)^2 - \sigma^2]}} = \varepsilon^{-\rho t}\frac{(p - \rho)}{\sqrt{(p^2 - \sigma^2)}}\varepsilon^{\rho t}$$

$$= \varepsilon^{-\rho t}\frac{p}{\sqrt{(p^2 - \sigma^2)}} = \varepsilon^{-\rho t} I_0(\sigma t)$$

or
$$i_0 = \frac{V}{Lv}\left(1 + \frac{G}{Cp}\right)\varepsilon^{-\rho t} I_0(\sigma t) \qquad (328)$$

CASE 1. With $G = 0$. Here $\rho = a = \sigma$, and we get

$$i_0 = \frac{V}{Lv} \varepsilon^{-\sigma t} I_0(\sigma t) \quad . \quad . \quad . \quad . \quad . \quad (329)$$

and

$$i_x = \frac{V}{Lv} \varepsilon^{-\alpha x} \varepsilon^{-\sigma t} I_0(\sigma t) \quad . \quad . \quad . \quad . \quad (330)$$

where

$$\alpha = \frac{1}{v} \sqrt{(p^2 + 2ap)} = \frac{1}{v} \sqrt{(p^2 + 2\sigma p)}$$

Shifting $\varepsilon^{-\sigma t}$ to the left, we have

$$i_x = \frac{V}{Lv} \varepsilon^{-\sigma t} . \varepsilon^{-\frac{x}{v}\sqrt{(p^2-\sigma^2)}} I_0(\sigma t) \quad . \quad . \quad . \quad (331)$$

Power expansion yields

$$i_x = \frac{V}{Lv} \varepsilon^{-\sigma t} \left\{ \left[1 + \frac{x^2}{v^2} \frac{p^2 - \sigma^2}{2!} + \frac{x^4}{v^4} \frac{(p^2 - \sigma^2)^2}{4!} \ldots \right] \right.$$
$$\left. - \left[1 + \frac{x^2}{v^2} \frac{p^2 - \sigma^2}{3!} + \frac{x^4}{v^4} \frac{(p^2 - \sigma^2)^2}{5!} \ldots \right] \left[\frac{x}{v} (p^2 - \sigma^2)^{\frac{1}{2}} \right] \right\} I_0(\sigma t) \quad (332)$$

Now

$$\frac{p}{\sqrt{(p^2 - \sigma^2)}} . (1) \doteqdot I_0(\sigma t)$$
$$\therefore \sqrt{(p^2 - \sigma^2)} I_0(\sigma t) \doteqdot p . (1) = 0$$

So that all terms in the second brackets are zero, and we are left with the first set of terms.

From the Bessel equation (see equation (454) of Chapter XI) given by

$$p^2 I_0(\sigma t) + \frac{1}{t} p I_0(\sigma t) - \sigma^2 I_0(\sigma t) = 0$$

we have $(p^2 - \sigma^2) I_0(\sigma t) = - \frac{p}{t} I_0(\sigma t) = - \frac{\sigma^2}{\sigma t} I_1(\sigma t)$

and, generally,

$$(p^2 - \sigma^2) I_m(\sigma t) = - (2m + 1) \frac{\sigma^2}{(\sigma t)^{m+1}} I_{m+1}(\sigma t)$$

On substituting, we get for (332)

$$i_x = \frac{V}{Lv} \varepsilon^{-\sigma t} \left[I_0(\sigma t) - \frac{\sigma^2 x^2}{v^2} . \frac{1}{2} \frac{I_1(\sigma t)}{\sigma t} \ldots \right] \quad . \quad (333)$$

This expression may be shown to reduce to

$$i_x = \frac{V}{Lv} \varepsilon^{-\sigma t} I_0 \left[\sigma \sqrt{\left(t^2 - \frac{x^2}{v^2} \right)} \right]_{t > x/v} \quad . \quad . \quad (334)$$

Comparing this result with equation (329) for i_0 shows that, given the value of current at the transmitter, we may write down the current at x by changing the argument of the Bessel to $\sqrt{\left(t^2 - \frac{x^2}{v^2}\right)}$. Heaviside gives several other methods of developing this equation.

CASE 2. With all constants present. We now have

$$V_x = V\varepsilon^{-\frac{x}{v}\sqrt{[(p+\rho)^2-\sigma^2]}}\,.\,(1) \quad . \quad . \quad . \quad . \quad (335)$$

and
$$I_x = vV\left(C + \frac{G}{p}\right)p.\frac{\varepsilon^{-\frac{x}{v}\sqrt{[(p+\rho)^2-\sigma^2]}}}{\sqrt{[(p+\rho)^2-\sigma^2}}\,.\,(1) \quad . \quad (336)$$

Now, by Campbell's table (*loc. cit.*) No. 866—

$$p\cdot\frac{\varepsilon^{-\frac{x}{v}\sqrt{[(p+\rho)^2-\sigma^2]}}}{\sqrt{[(p+\rho^2)-\sigma^2]}}\,.\,(1) \doteqdot \varepsilon^{-\rho t}I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right] \quad (337)$$

So

$$I_x = V\sqrt{\left(\frac{C}{L}\right)}\varepsilon^{-\rho t}I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]$$
$$+ vGV_0\int_{x/v}^{t}\varepsilon^{-\rho\lambda}I_0\left[\sigma\left(\lambda^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]d\lambda \quad . \quad (338)$$

and
$$V_x = V\left\{\varepsilon^{-\rho x/v} + \frac{\sigma x}{v}\int_{x/v}^{t}\varepsilon^{-\rho\lambda}\frac{I_0\left[\sigma\left(\lambda^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]}{\sqrt{\left(\lambda^2-\frac{x^2}{v^2}\right)}}d\lambda\right\}_{t>x/v} \quad (339)$$

3. **Finite Line with Terminal Impedances (Wave Solution).** Let a steady voltage E be impressed on the line through an impedance Z_1, and let the line be closed at $x = l$ through an impedance Z_2. On substituting in the general line equations—

$$V_x = A\varepsilon^{\alpha x} + B\varepsilon^{-\alpha x},$$

and
$$I_x = \frac{1}{Lv}[B\varepsilon^{-\alpha x} - A\varepsilon^{\alpha x}]\sqrt{\left[\frac{p+2b}{p+2a}\right]}$$

we get
$$\frac{Z_2}{Z_{(p)}}[B\varepsilon^{-\alpha l} - A\varepsilon^{\alpha l}] = A\varepsilon^{\alpha l} + B\varepsilon^{-\alpha l} \quad . \quad . \quad . \quad (340)$$

and
$$A + B = E - \frac{Z_1}{Z_{(p)}}(B - A) \quad . \quad . \quad (341)$$

where $\dfrac{1}{Z_{(p)}} = \dfrac{1}{Lv}\sqrt{\left[\dfrac{p + 2b}{p + 2a}\right]}$

With $\dfrac{Z_1}{Z_{(p)}} = \rho_1$ and $\dfrac{Z_2}{Z_{(p)}} = \rho_2$, it is easily shown that

$$A = \frac{(\rho_2 - 1)\varepsilon^{-2\alpha l}}{(\rho_1 + 1)(\rho_2 + 1) - (\rho_1 - 1)(\rho_2 - 1)\varepsilon^{-2\alpha l}} E$$

$$B = \frac{(\rho_2 + 1)}{(\rho_1 + 1)(\rho_2 + 1) - (\rho_1 - 1)(\rho_2 - 1)\varepsilon^{-2\alpha l}} E$$

On substituting, we get

$$I_x = \frac{E}{Z_1 + Z_{(p)}}\left[\frac{\varepsilon^{-\alpha x} - \dfrac{\rho_2 - 1}{\rho_2 + 1}\varepsilon^{-\alpha(2l - x)}}{1 - \dfrac{\rho_1 - 1}{\rho_1 + 1}\cdot\dfrac{\rho_2 - 1}{\rho_2 + 1}\varepsilon^{-2\alpha l}}\right] \qquad . \quad . \quad (342)$$

Denoting $\dfrac{\rho_1 - 1}{\rho_1 + 1}$ by $-\mu_1$ and $\dfrac{\rho_2 - 1}{\rho_2 + 1}$ by $-\mu_2$, we have

$$I_x = \frac{E}{Z_1 + Z_{(p)}}\left[\frac{\varepsilon^{-\alpha x} + \mu_2\varepsilon^{-\alpha(2l - x)}}{1 - \mu_1\mu_2\varepsilon^{-2\alpha l}}\right] \qquad . \quad . \quad . \quad . \quad . \quad (343)$$

On expanding we get

$$I_x = \frac{E}{Z_1 + Z_{(p)}}[\varepsilon^{-\alpha x} + \mu_2\varepsilon^{-\alpha(2l-x)} + \mu_1\mu_2\varepsilon^{-\alpha(2l+x)} + \mu_1\mu_2{}^2\varepsilon^{-\alpha(4l-x)} + \ldots] \quad . \quad (344)$$

For an infinite line fed through an impedance Z_1 at $x = 0$ the current at x is given operationally by

$$I_x = \frac{E}{Z_1 + Z_{(p)}}\varepsilon^{-\alpha x}$$

So the first term in (344) is the current at x due to E impressed at $x = 0$. Similarly the second term gives the current at $(2l - x)$, and so on. The first term gives the effect of the direct wave, while the second gives the effect of the wave reflected from the far end, and the third that of the wave reflected from the near end. The above equation is thus the sum of direct and reflected waves which are effective at times $t = x/v$, $t = (2l - x)/v$, $t = (2l + x)/v$, etc. Thus if we require to know the current at a time less than $(2l + x)/v$, we should take into account only the first two terms of equation (344).

4. **The Voltage after Infinite Time.** In practice the usual problem is to find the voltage after an infinite time. Let the voltage E be impressed on a distortionless line at C. The line will attenuate as ε^{-qx} and have reflection coefficients α and β at A and B respectively (Fig. 29). For a voltage wave moving

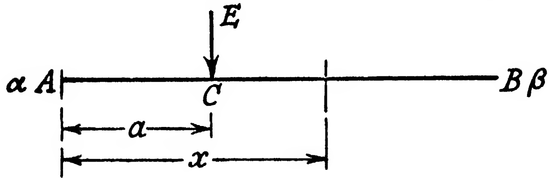

FIG. 29

to the right we have that, at x, the wave is $v_1 = \frac{1}{2}E\varepsilon^{-q(x-a)}$. At B the value of the wave will be $\frac{1}{2}E\varepsilon^{-q(l-a)}$, and the reflected wave is $\beta \cdot \frac{1}{2}E\varepsilon^{-q(l-a)}$. When this wave passes x, its value is

$$\beta \cdot \tfrac{1}{2}E\varepsilon^{-q(l-a)} \cdot \varepsilon^{-q(l-x)} = \beta \cdot \tfrac{1}{2}E\varepsilon^{-q(2l-a-x)}$$

So at A it will have a value of $\beta \cdot \frac{1}{2}E\varepsilon^{-q(2l-a)}$, and the reflected wave will be $\alpha\beta\frac{1}{2}E\varepsilon^{-q(2l-a)}$. This is attenuated to

$$\alpha\beta\tfrac{1}{2}E\varepsilon^{-q(2l-a)} \cdot \varepsilon^{-qx}$$

at x. Now, at x, the value of the voltage will be the sum of the voltages, thus

$$V_x = \tfrac{1}{2}E\left[\varepsilon^{-q(x-a)} + \beta\varepsilon^{-q(2l-a-x)} + \alpha\beta\varepsilon^{-q(2l-a+x)} + \ \ldots\right]$$

$$= \tfrac{1}{2}E\left[\frac{\varepsilon^{-q(x-a)}\left[1 + \beta\varepsilon^{-2q(l-x)}\right]}{1 - \alpha\beta\varepsilon^{-2ql}}\right] \quad . \quad . \quad . \quad (345)$$

In a similar manner the waves travelling to the *left* will give a voltage

$$-\tfrac{1}{2}E\left[\frac{\alpha\varepsilon^{-q(x+a)}\left[1 + \beta\varepsilon^{-2q(l-x)}\right]}{1 - \alpha\beta\varepsilon^{-2ql}}\right] \quad . \quad . \quad . \quad (346)$$

So that the total voltage at A is given by the sum of (345) and (346). We thus have

$$V = \tfrac{1}{2}E\left[\frac{\left[\varepsilon^{qa} - \alpha\varepsilon^{-qa}\right]\left[\varepsilon^{q(l-x)} + \beta\varepsilon^{-q(l-x)}\right]}{\varepsilon^{ql} - \alpha\beta\varepsilon^{-ql}}\right] \quad . \quad . \quad (347)$$

A convenient method of following through any voltage impulse is by means of a lattice diagram shown in Fig. 30. This has been constructed for the pulse travelling to the right only. The pulse travelling to the left would need to be superimposed on the diagram. From a knowledge of the length of

line the times at which the various pulses arrive could be determined, and so the voltage at any point could be found by the addition of the pulses at their correct times. This is discussed in Section 9 of this chapter.

From equation (347), by substituting different values of α and β we are able to write down the values of the voltage at x.

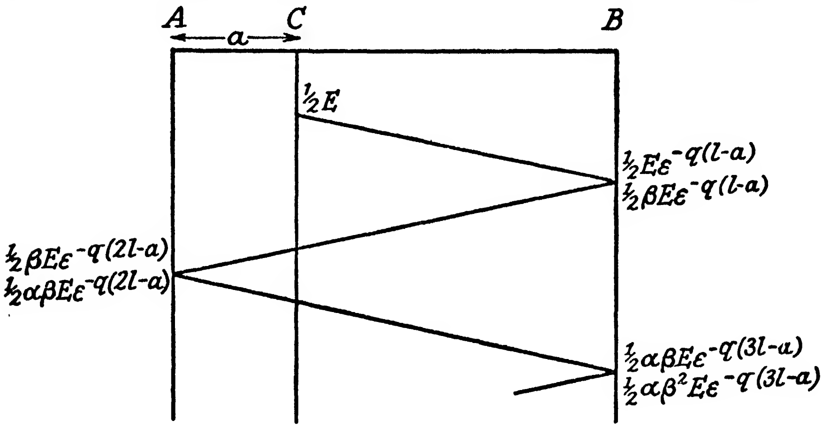

FIG. 30

For example, with a line open at both ends we have $\alpha = \beta = +1$, and (347) reduces to

$$V = E\frac{\sinh qa \,.\, \cosh q(l-x)}{\sinh ql} \quad . \quad . \quad (348)$$

When both ends are earthed, then $\alpha = \beta = -1$ and the equation becomes

$$V = E\frac{\cosh qa \,.\, \sinh q(l-x)}{\sinh ql} \quad . \quad . \quad (349)$$

With an earth at A and the line open at B we have that $\alpha = -1, \beta = +1$, and so

$$V = E\frac{\cosh qa \,.\, \cosh q(l-x)}{\cosh ql} \quad . \quad . \quad (350)$$

These expressions give the steady-state solutions and confirm results given in Section 3 of Chapter VII.

5. **Efficiency of Reflectors.** In Section 3 we have seen that the reflection coefficients at the far and near ends of the line are given by $\mu_2 = \dfrac{Z_{(p)} - Z_2}{Z_{(p)} + Z_2}$ and $\mu_1 = \dfrac{Z_{(p)} - Z_1}{Z_{(p)} + Z_1}$ respectively,

where $Z_{(p)} = \sqrt{(Z/Y)}$ is the operational characteristic impedance of the line.

For steady-state conditions in a line terminated by an impedance Z_1 at $x = l$ we have the following equations—

$$E_x = \tfrac{1}{2}E_R\left[1 + \frac{Z_0}{Z_1}\right]\varepsilon^{\alpha(l-x)} + \tfrac{1}{2}E_R\left[1 - \frac{Z_0}{Z_1}\right]\varepsilon^{-\alpha(l-x)}$$

$$I_x = \sqrt{\left(\frac{Y}{Z}\right)}\left[\tfrac{1}{2}E_R\left(1 + \frac{Z_0}{Z_1}\right)\varepsilon^{\alpha(l-x)} - \tfrac{1}{2}E_R\left(1 - \frac{Z_0}{Z_1}\right)\varepsilon^{-\alpha(l-x)}\right]$$

The incident voltage is

$$\tfrac{1}{2}E_R\left(1 + \frac{Z_0}{Z_1}\right) = v_1 \quad . \quad . \quad . \quad (351)$$

and the reflected voltage is

$$\tfrac{1}{2}E_R\left(1 - \frac{Z_0}{Z_1}\right) = v_2 \quad . \quad . \quad . \quad (352)$$

At $x = l$ we have $E_R = v_1 + v_2$. The current equation may be written as $i_x = Y_0[v_1\varepsilon^{\alpha(l-x)} - v_2\varepsilon^{-\alpha(l-x)}]$ so that the incident current wave is

$$i_1 = Y_0v_1 \quad . \quad . \quad . \quad . \quad (353)$$

and the reflected is

$$i_2 = -Y_0v_2 \quad . \quad . \quad . \quad . \quad (354)$$

The efficiency of the reflector is

$$\frac{v_2}{v_1} = \frac{Z_1 - Z_0}{Z_1 + Z_0} = -\frac{i_2}{i_1} \quad . \quad . \quad . \quad (355)$$

We now apply these results to some simple line conditions as follows.

(i) Line Open. In this instance Z_1 is infinite, and Z_0 is negligibly small. From equations (351) and (352) we get that the incident and the reflected voltages are each equal to $\tfrac{1}{2}E_R$ and are both of the same sign. We may write these as $v_1 = \tfrac{1}{2}I_RZ_1$ and $v_2 = \tfrac{1}{2}I_RZ_1$. From equations (353) and (354) we get that the currents are $i_1 = \tfrac{1}{2}E_RY_0$ and $i_2 = -\tfrac{1}{2}E_RY_0$, so that the voltage at the end of the line is the sum of the incident and reflected volts, and the current is zero.

(ii) Line on Short Circuit. Here $Z_1 = 0$, so that $Y_1 = \infty$. The value of the incident and reflected voltage waves will be given by $v_1 = \tfrac{1}{2}Z_0I_R$ and $v_2 = -\tfrac{1}{2}Z_0I_R$. The current waves are

given by $i_1 = \frac{1}{2}Y_0E_R$ and $i_2 = \frac{1}{2}Y_0E_R$. So, for a line on short circuit, the voltage at the far end is given by

$$E_R = v_1 + v_2 = 0$$

and the receiver current is

$$I_R = 2i_1 = E_RY_0$$

(iii) HARMONICS IN WAVE. For a wave of composite frequencies, in the case of a non-dissipative line, the reflection coefficient is

$$\beta_n = \frac{X_{1n} - X_{0n}}{X_{1n} + X_{0n}}$$

and thus is independent of frequency. Thus no distortion occurs in the reflections. When resistance effects are present, then we can no longer consider that $\beta_1 = \beta_2 = \ldots = \beta_n$, and distortion will occur. The generalized problem becomes very complex.

6. **Series Circuit.** For an overhead line and cable in series, by the method of the last section we have $\beta = \frac{v_2}{v_1} = \frac{Z_c - Z_l}{Z_c + Z_l}$ as the reflection factor into the cable. When v_1 is the incident volts, the voltage at the junction becomes

$$V = v_1 + v_1\beta = v_1\left[\frac{2Z_c}{Z_c + Z_l}\right]$$

This is the voltage transmitted into the cable. The voltage reflected into the overhead line is

$$2v_2\left[\frac{Z_c}{Z_c - Z_l}\right]$$

The incident current is

$$i_1 = \frac{v_1}{Z_l} = V\frac{Z_c + Z_l}{2Z_cZ_l}$$

and the reflected current is

$$i_2 = -\frac{v_2}{Z_l} = V \cdot \frac{Z_c - Z_l}{2Z_cZ_l}$$

The current transmitted into the cable will be

$$i_1 - i_2 = V \cdot \frac{2Z_l}{2Z_cZ_l} = \frac{2v_1}{Z_c + Z_l}$$

For a wave travelling from cable into line we have

$$\frac{v_2'}{v_1'} = \frac{Z_l - Z_c}{Z_l - Z_c}$$

and $$V = v_1' + v_2'$$

So that the incident voltage becomes

$$v_1' = \frac{V}{2}\left[\frac{Z_c + Z_l}{Z_l}\right]$$

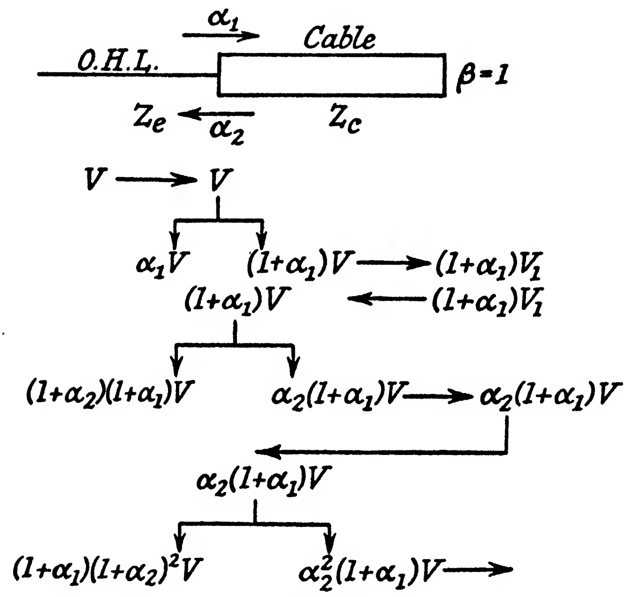

FIG. 31

and the voltage reflected back into the cable is

$$v_2' = \frac{V}{2}\left[\frac{Z_l - Z_c}{Z_l}\right]$$

Similarly the incident current wave is

$$i_1' = \frac{v_1'}{Z_c} = \frac{V}{2}\frac{Z_c + Z_l}{Z_c Z_l}$$

and the reflected current is

$$i_2' = \frac{V}{2} \cdot \frac{Z_l - Z_c}{Z_c Z_l}$$

So the transmitted current is given by $\dfrac{2v_1'}{Z_c + Z_l}$.

Summarizing, we have the following coefficients—

From Overhead Line to Cable

Reflection coefficient $\alpha_1 = \dfrac{Z_l - Z_c}{Z_l + Z_c}$; transmission coefficient $= 1 + \alpha_1$.

From Cable to Overhead Line

Reflection coefficient $\alpha_2 = \dfrac{Z_c - Z_l}{Z_l + Z_c}$; transmission coefficient $= 1 + \alpha_2$.

We are now able to study the case of a voltage fed into an open-circuited line, i.e. when $\beta = 1$: as shown in Fig. 31, the line losses are neglected. The voltage on the cable builds up to

$$2(1 + \alpha_1)V[1 + \alpha_2 + \alpha_2^2 + \ldots] = 2(1 + \alpha_1)V \cdot \frac{1}{1 - \alpha_2}$$

As $\alpha_1 = -\alpha_2$, the final voltage at the junction is $2E$.

7. **Series-parallel Circuits.** Fig. 32 shows a typical arrangement consisting of a line of surge impedance Z feeding on to a

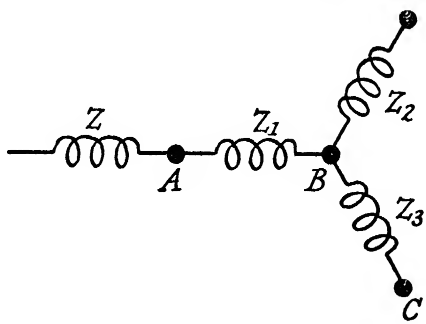

Fig. 32

transformer of lumped impedance Z_1. From the other side, lines of surge impedances Z_2 and Z_3 are taken. The impedance at A looking into the network is

$$Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = Z_0 \text{ say.}$$

The reflection coefficient is

$$\beta = \frac{Z_0 - Z}{Z_0 + Z}$$

If v_1 is the incident voltage, then the reflected voltage is

$$v_2 = v_1\left(\frac{Z_0 - Z}{Z_0 + Z}\right)$$

The potential at A will be

$$v_1 + v_2 = v_1\frac{2Z_0}{Z_0 + Z}$$

The current through Z will be

$$\frac{v_1}{Z_0}\cdot\frac{2Z_0}{Z_0 + Z}$$

The potential at B will be equal to the voltage at A minus the potential drop in Z_1 and so will be

$$V_B = v_1\frac{2Z_0}{Z_0 + Z} - \frac{v_1}{Z_0}\,\frac{2Z_0}{Z_0 + Z}\,Z_1 = 2v_1\frac{Z_0 - Z_1}{Z_0 + Z}$$

Further, the current through Z_3 will be given by

$$i_3 = \frac{2v_1}{Z_3}\,\frac{Z_0 - Z_1}{Z_0 + Z} = 2v_1\left(\frac{Z_2}{Z_2 + Z_3}\cdot\frac{1}{Z_0 + Z}\right)$$

The voltage drop across the whole is given by

$$iZ_1 + i_3Z_3 = \frac{2v_1}{Z + Z_0}\left(\frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2Z_3}\right)$$

To fix ideas let us consider a line of surge impedance Z_1 connected to an inductive coil which is in series with a line of

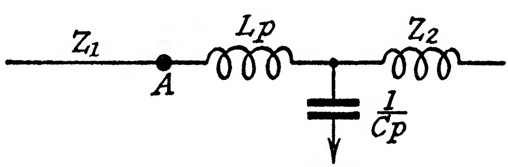

FIG. 33

surge impedance Z_2, the junction being earthed through a capacity C (Fig. 33). With the above notation we get

$$Z_0 = Lp + \frac{Z_2}{1 + Z_2Cp} = \frac{Z_2LCp^2 + Lp + Z_2}{1 + Z_2Cp}$$

and the reflection coefficient at A is

$$\beta = \frac{v_2}{v_1} = \frac{Z_0 - Z_1}{Z_0 + Z_1}$$

The potential at A is $v\,(1+\beta)$ and the current through Lp is $\frac{v_1}{Z_0}\,(1+\beta)$. The voltage at B is

$$v_1\,(1+\beta) - \frac{v_1}{Z_0}\,(1+\beta)\,Lp = v_1\,(1+\beta)\left(1-\frac{Lp}{Z_0}\right)$$

the current through the condenser is

$$i_c = Cpv_1\,(1+\beta)\left(1-\frac{Lp}{Z_0}\right)$$

So that the current into the line is $I - i_c$.

It is seen that each individual equation is in operational form; for evaluation, the impressed voltage wave form must be known.

8. **Effect of Lightning Stroke.** We will consider a lightning stroke on a line terminated by an inductive coil. The consensus of opinion seems to be that the voltage surge due to lightning may be represented as $E\varepsilon^{-\alpha t}$. Here v_1, the incident voltage, is equal to $E\varepsilon^{-\alpha t}\cdot(1)$, and the reflected voltage is

$$v_2 = E\left(\frac{Lp - Z_0}{Lp + Z_0}\right)\varepsilon^{-\alpha t}\cdot(1)$$

By shifting, we get

$$v_2 = E\varepsilon^{-\alpha t}\left[\frac{L\,(p-\alpha) - Z_0}{L\,(p-\alpha) + Z_0}\right]\cdot(1)$$

The expansion theorem gives

$$v_2 = \frac{E\varepsilon^{-\alpha t}}{\alpha L - Z_0}\left[(\alpha L + Z_0) - 2Z_0\varepsilon^{(\alpha - Z_0/L)t}\right]$$

The potential at B is given by

$$\frac{E\varepsilon^{-\alpha t}}{\alpha L - Z_0}\left[2\alpha L - 2Z_0\varepsilon^{(\alpha - Z_0/L)t}\right]$$

and the current through the inductance is

$$\frac{2v_1}{Z_0 + Lp}\cdot(1) = \varepsilon^{-\alpha t}\frac{2E}{Z_0 + (Lp - a)}\cdot(1)$$

$$= \frac{2E}{Z_0 - \alpha L}\,\varepsilon^{-\alpha t}\left[1 + \varepsilon^{(\alpha - Z_0/L)t}\right]$$

If instead of an inductive earth, a capacitative earth had been used, then the current through the condenser would have been

$$\frac{2v_1}{1 + Z_0Cp} Cp \,.\, (1)$$

or
$$i_c = \frac{2EC\varepsilon^{-\alpha t}}{\alpha Z_0C - 1}\left[\alpha + \frac{1}{Z_0C}\,\varepsilon^{(\alpha - 1/Z_0C)t}\right]$$

9. **Overhead Line Terminated by a Cable.** A length of cable is frequently used to absorb surge voltage and so protect machines connected to the system. In the cable system shown

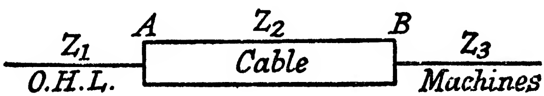

FIG. 34

in Fig. 34 we will investigate the voltage occurring at the cable connection. The effect on the wave form is neglected.

NEGLECTING ATTENUATION AND TIME EFFECTS. THE VOLTAGE AT B. When an impulse voltage E is applied at A, the voltage transmitted to the cable is $\frac{2Z_2}{Z_1 + Z_2}E$. This voltage is reflected at B, and the voltage at B is given by

$$\frac{2Z_2}{Z_1 + Z_2}\left(1 + \frac{Z_3 - Z_2}{Z_3 + Z_2}\right)E$$

and the reflected part is

$$\frac{2Z_2}{Z_1 + Z_2} \cdot \frac{Z_3 - Z_2}{Z_3 + Z_2} \cdot E$$

On the reflected voltage reaching A it is reflected again, and this component is

$$\frac{2Z_2}{Z_1 + Z_2} \cdot \frac{Z_3 - Z_2}{Z_3 + Z_2} \cdot \frac{Z_1 - Z_2}{Z_1 + Z_2} E = \frac{2Z_2}{Z_1 + Z_2} M$$

where $M = \frac{Z_1 - Z_2}{Z_1 + Z_2} \cdot \frac{Z_3 - Z_2}{Z_3 + Z_2} E.$

Earthing both ends of the system, we can concentrate on the

wave within the cable. The value of the wave is now $\frac{2Z_2}{Z_1 + Z_2} M$ moving towards B. At B the reflected voltage is

$$\frac{2Z_2}{Z_1 + Z_2} M \frac{Z_3 - Z_2}{Z_3 + Z_2}$$

and the total voltage at B is

$$\frac{2Z_2}{Z_1 + Z_2}\left(1 + \frac{Z_3 - Z_2}{Z_3 + Z_2}\right)M$$

Similarly for the next stage, the reflected voltage at A is given by

$$\frac{2Z_2}{Z_1 + Z_2} \cdot \frac{Z_3 - Z_2}{Z_3 + Z_2} \cdot \frac{Z_1 - Z_2}{Z_1 + Z_2} M = \frac{2Z_2}{Z_1 + Z_2} M^2$$

and the voltage at B is

$$\frac{2Z_2}{Z_1 + Z_3} M^2 \left(1 + \frac{Z_3 - Z_2}{Z_3 + Z_2}\right)$$

So the total voltage at B is given by

$$E_B = \frac{2Z_2}{Z_1 + Z_2}\left(1 + \frac{Z_3 - Z_2}{Z_3 + Z_2}\right)(1 + M + M^2 + \ldots)$$

$$= E\left(1 + \frac{Z_2 - Z_1}{Z_2 + Z_1} + \frac{Z_3 - Z_2}{Z_3 + Z_2} - m\right)\frac{1}{1 - m}$$

where $mE = M$ and $m < 1$.

When $Z_1 = Z_2 = Z_3$, then $E_B = E$.

Considering Time Effects. The Voltage at B. Let the time for a wave to travel twice the length of the cable be T: reckoning time from the arrival of the first wave at B, we have that the initial voltage at B is

$$E \frac{2Z_2}{Z_1 + Z_2}\left(1 + \frac{Z_3 - Z_2}{Z_3 + Z_2}\right)\varepsilon^0 = EP\varepsilon^0 \text{ (say)}$$

The second arrival will be

$$EmP\varepsilon^{-\alpha T}$$

The third will be

$$Em^2P\varepsilon^{-2\alpha T}$$

and so on.

If we sum for n steps, i.e. for nT seconds, the effective value of the first voltage at B (see Fig. 35) is given by

$$E \,.\, P \,.\, \varepsilon^{-\alpha T(n-1)}$$

the second by

$$EPm\varepsilon^{-\alpha T(n-2)}$$

the third by

$$EPm^2\varepsilon^{-\alpha T(n-3)}$$

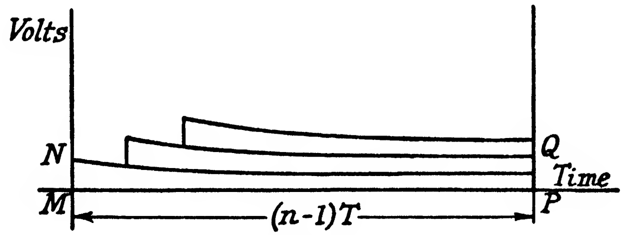

FIG. 35

and so on. So that the total voltage is

$$E_B = PE\varepsilon^{-(n-1)\alpha T}\left[\frac{1-(m\varepsilon^{\alpha T})^n}{1-m\varepsilon^{\alpha T}}\right]$$

With $\alpha = 0$ we get

$$E_B = PE\left(\frac{1-m^n}{1-m}\right)$$

As m is less than unity, the value of E may be several times that of E_B. For $Z_1 = Z_2 = Z_3$ we get $E_B = E$.

FOR THE END A. The initial voltage at A is

$$E\left(1+\frac{2Z_2}{Z_1+Z_2}\right)\varepsilon^0$$

This pulse is partially transmitted and partially reflected. The transmitted part is reflected again from B, and on reaching A the voltage is given by

$$E \,.\, \frac{2Z_2}{Z_1+Z_2} \cdot \frac{Z_3-Z_2}{Z_3+Z_2}\left(\quad + \frac{Z_1-Z_2}{Z_1+Z_2}\right)$$

or

$$E \,.\, \frac{2Z_2}{Z_1+Z_2}\left(\frac{Z_3-Z_2}{Z_3+Z_2}+m\right)$$

The reflected part moves to B as $E\,.\,\dfrac{2Z_2}{Z_1 + Z_2}\,m.$

So that the total voltage at A is given by

$$E\,.\,\frac{2Z_2}{Z_1 + Z_2}\left(m\,\frac{Z_3 - Z_2}{Z_3 + Z_2} + m^2\right)$$

Summing for n steps, we have

$$E_A = E\left[\varepsilon^{-(n-1)\alpha T} + P\varepsilon^{-(n-1)\alpha T} + P\left(\frac{Z_3 - Z_2}{Z_3 + Z_2} + m\right)\varepsilon^{-(n-2)\alpha T} \ldots\right]$$

$$= E\left\{\varepsilon^{-(n-1)\alpha T} + P\varepsilon^{-(n-1)\alpha T}\left[\frac{1 - (m\varepsilon^{\alpha T})^{n-1}}{1 - m\varepsilon^{\alpha T}}\right] + P\left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)\varepsilon^{-(n-2)\alpha T}\left[\frac{1 - (m\varepsilon^{\alpha T})^{n-2}}{1 - m\varepsilon^{\alpha T}}\right]\right\}$$

where $P = \dfrac{2Z_2}{Z_1 + Z_2}$.

With $\alpha = 0$, we get

$$E_A = E\left[1 + \frac{2P}{1 - m}\left(\frac{Z_3}{Z_2 + Z_3}\right) - m^{n-1}\left(\frac{Z_1}{Z_2 + Z_3}\right)\right]$$

And when $Z_1 = Z_2 = Z_3$ this reduces to $E_A = 2E$.

10. **A Condenser Discharging into a Short Length of Line on Open Circuit.** Let Z be the impedance of the line and $1/Cp$ the operational impedance of the condenser. Then, at the condenser terminals, we have the reflection coefficient as

$$\frac{1/Cp - Z}{1/Cp + Z} = \frac{\alpha - p}{\alpha + p} = m$$

where $\alpha = 1/CZ$. The transmission coefficient will be

$$1 + \frac{\alpha - p}{\alpha + p} = \frac{2\alpha}{\alpha + p} = n$$

The condenser discharge may be formulated as $E\varepsilon^{-\beta t}$, and, during the passage of the voltage along the line, let it be

attenuated to $E\varepsilon^{-\beta t} . \varepsilon^{-\alpha x} = ae._1$ The action may be summarized as follows—

Condenser		Far End of Line
Charge	*Discharge*	
E . . . e_1		$\left.\begin{matrix} ae_1 \\ ae_1 \end{matrix}\right\}$
a^2e_1		
na^2e_1	ma^2e_1	$\left.\begin{matrix} ma^3e_1 \\ ma^3e_1 \end{matrix}\right\}$
ma^4e_1		$\left.\begin{matrix} m^2a^5e_1 \\ m^2a^5e_1 \end{matrix}\right\}$
$n^2a^4e_1$	$m^2a^4e_1$	
etc.		etc.

Then by summation we have that the voltage at the end of the line is

$$2\,[ae_1 + ma^3e_1 + m^2a^5e_1 \;.\;.\;.] = \frac{2ae_1}{1 - m\alpha^4}$$

The residual voltage on the condenser is

$$E - e_1 - a^2me_1 - a^4m^2e_1, \;.\;.\;. + a^2ne_1 + a^4n^2e_1 + \;.\;.\;.$$

$$= E - e_1 - a^2e_1\left[\frac{m - n}{(1 - a^2n)\,(1 - a^2m)}\right]$$

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CHAPTER IX

EDDY CURRENT EFFECTS

1. **Fundamental Considerations.** Closely allied in physical conception with the diffusion of electricity along a wire is the penetration of flux within an iron plate and problems such as the electrical resistance of copper bars to currents of high frequency. The fundamental equations of these problems are based on Faraday's and Ampère's laws. These are derived as follows—

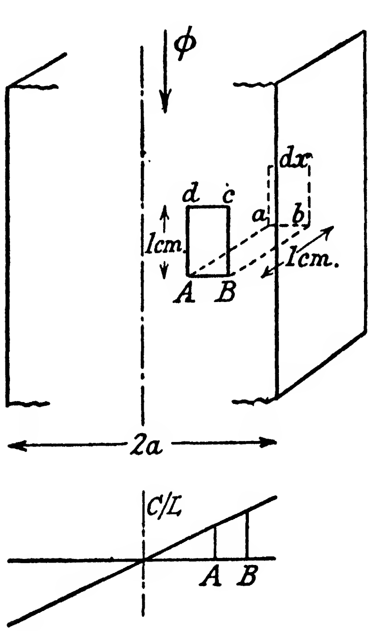

Fig. 36

FARADAY'S LAW. For an iron plate as shown in Fig. 36, let the flux be acting downwards as shown. For an elementary area, of length 1 cm. and width dx at right angles to the plane of the flux, the total flux through it will be $B \, . \, dx$ where B is the flux density. The e.m.f. induced in this circuit $ABba$ is $-\dfrac{\partial}{\partial t} B \, . \, dx$. If the current density at A is σ, then at B it will be $\sigma + \dfrac{\partial \sigma}{\partial x} dx$. The potential difference round this element will then be

$$\left(\sigma + \frac{\partial \sigma}{\partial x} dx\right)\rho - \sigma\rho = \rho \frac{\partial \sigma}{\partial x} dx$$

where ρ is the resistivity of the iron and the voltage drops across the ends AB and ab are negligible. So we have that

$$-\frac{\partial}{\partial t}(Bdx) = \rho \frac{\partial \sigma}{\partial x} dx \quad . \quad . \quad . \quad (356)$$

AMPÈRE'S LAW. For an elementary area $ABcd$ parallel with the plane of the paper, of magnitude 1 cm. $\times$ dx and with a current density of σ, the total current through the area is $\sigma \, . \, dx$. Hence the work done in taking unit magnetic pole once

round this area is $4\pi\sigma \, . \, dx$. When H is the magnetic force at A and $H + \frac{\partial H}{\partial x}dx$ that at B, the work done is

$$H - \left[H + \frac{\partial H}{\partial x}dx\right] = -\frac{\partial H}{\partial x}dx$$

So that we have
$$4\pi\sigma dx = -\frac{\partial H}{\partial x}dx$$
$$= -\frac{1}{\mu}\frac{\partial B}{\partial x}dx \quad . \quad . \quad (357)$$

where ρ and μ are in e.m. units.

2. **Resistance of a Round Conductor.** On neglecting the field set up by the eddy currents, the resistance of a conductor may be found approximately by the following method, due to Moullin. The complete solution is given in Chapter XI. In a conductor of length l and cross-sectional area πa^2, let the current be $I_0 \sin \omega t$. For low-frequency alternating current, the current density will be uniform and equal to $\sigma_0 = I_0/\pi a^2$. At high frequencies the distribution is not uniform, because the induced e.m.f. is sufficient in magnitude to produce distortion. The density will be given in the form

$$\sigma = \sigma_0 \pm \sigma_e$$

where σ_e is the density of the eddy currents.

As the eddy currents do not appear in the external circuit, then $\Sigma\sigma_e = 0$, i.e. the eddy currents must flow in opposite directions on each side of some radius r_0. Let us consider a section $abcd$ at right angles to the magnetic flux with bc on the neutral radius r_0 as in Fig. 37. As the magnetic field strength H within the conductor increases uniformly from zero at the centre to $(2I_0/a) \sin \omega t$ at the external surface, then the flux through the rectangle will be

$$ad\int_r^{r_0} \frac{r}{a}Hdr = ad\int_r^{r_0} \frac{2I_0}{a^2}r \sin \omega t dr = ad(r_0^2 - r^2)\frac{I_0 \sin \omega t}{a^2}$$

So, Faraday's Law may be written,

$$\rho\sigma_e = -\frac{d}{dt}(r_0^2 - r^2)\frac{I_0 \sin \omega t}{a^2}$$

But $I_0 = \pi a^2\sigma_0$; so

$$\sigma_e = \frac{1}{\rho}\pi\omega(r^2 - r_0^2)\sigma_0 \cos \omega t \quad . \quad . \quad . \quad (358)$$

As $\int_0^a 2\pi r\sigma_e dr = 0$, so

$$\int_0^a \frac{1}{\rho} 2\pi^2 \omega r(r^2 - r_0^2)\sigma_0 \cos \omega t dw = 0$$

On eliminating factors independent of r we have

$$\frac{a^4}{4} - \frac{a^2 r_0^2}{2} = 0 \quad . \quad . \quad . \quad . \quad (359)$$

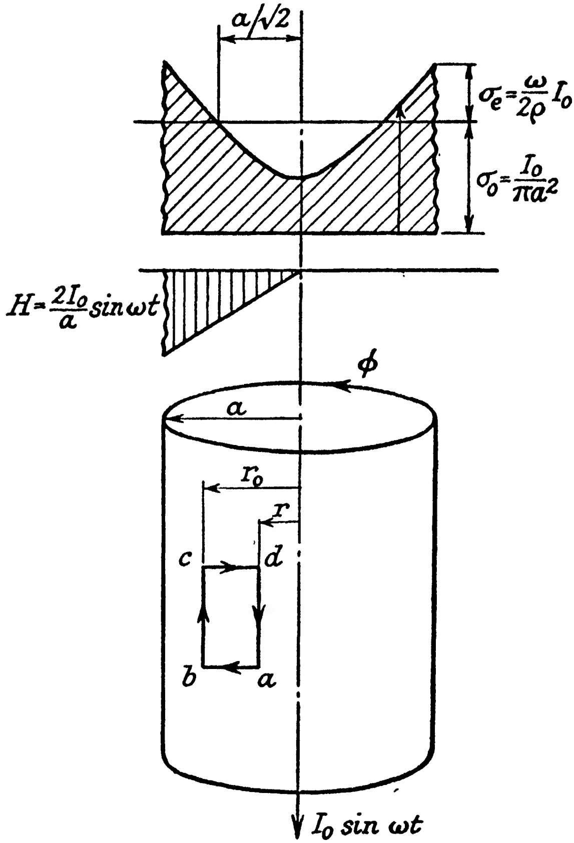

FIG. 37

$\therefore\ r_0 = a/\sqrt{2}$ as the radius of the neutral cylinder. The eddy currents inside and outside this cylinder flow in opposite directions.

So that from equation (358)

$$\sigma_e = \frac{\pi\omega}{\rho} (r^2 - \tfrac{1}{2}a^2)\sigma_0 \cos \omega t$$

The resultant current density will be given by

$$\sigma = \frac{I_0}{\pi a^2}\sin\omega t + \frac{\omega}{\rho}\left(\frac{r^2}{a^2} - \frac{1}{2}\right)I\cos\omega t \qquad . \quad (360)$$

i.e. σ is the sum of a uniform density $\sigma_0 = I_0/\pi a^2$ together with a density σ_e which is a function of the radius: where $r < a/\sqrt{2}$ the eddy current is in phase opposition to σ_0, and where $r > a/\sqrt{2}$ it is in phase therewith. The current is crowded towards the outer surface, giving a skin effect.

Resistance. The mean power loss at low frequencies is given by $\frac{1}{2}RI_0^2$; where $R = \rho l/\pi a^2$. At high frequencies the mean loss of power is defined as $\frac{1}{2}R_eI_0^2$, where R_e is the effective resistance. After some calculation we obtain that the mean power is

$$\tfrac{1}{2}RI_0^2\left[1 + \frac{2\pi^2\omega^2}{a^4\rho^2}\int_0^a (r^2 - \tfrac{1}{2}a^2)^2 r\,dr\right]$$

So
$$\tfrac{1}{2}R_eI_0^2 = \tfrac{1}{2}RI_0^2\left(1 + \frac{2\pi^2\omega^2}{\rho^2}\cdot\frac{a^4}{24}\right)$$

and
$$\frac{R_e}{R} = 1 + \frac{\pi^4 f^2 a^4}{3\rho^2} \qquad . \quad . \quad . \quad . \quad (361)$$

3. **Eddy Currents in Laminations.** We will assume that the permeability is constant for the range of flux densities encountered and that the hysteretic angle is negligible. With $B = \mu H$ we have, for Fig. 36—

$$\rho\cdot\frac{d\sigma}{dx} = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t}$$

$$4\pi\sigma = -\frac{\partial H}{\partial x}$$

Combining these, we get

$$\frac{4\pi\mu}{\rho}\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} \qquad . \quad . \quad (362)$$

With $H = H_0\varepsilon^{j\omega t}$ we have

$$\frac{\partial^2 H}{\partial x^2} = (4\pi\mu/\rho)\,j\omega H_0\varepsilon^{j\omega t} = m^2 H_0\varepsilon^{j\omega t}$$

where $m^2 = (4\pi\mu\omega/\rho)j$ or $m = [\sqrt{(2\pi\mu\omega/\rho)}]\,(1 + j)$.

This equation is of the same type as that for a transmission line, and is integrated as

$$H = A\varepsilon^{(1+j)mx} + B\varepsilon^{-(1+j)mx} \qquad . \quad . \quad (363)$$

where A and B are constants determinate from boundary conditions.

For a plate of thickness $2a$ the centre line may be regarded as an axis of symmetry. So that quantities at $x = a$ and $x = -a$ must be identical. Thus $A = B$, and if H_0 is the amplitude of H at the surface we get

$$A = \frac{H_0}{\varepsilon^{(1+j)ma} + \varepsilon^{-(1+j)ma}}$$

so
$$H_x = H_0\left(\frac{\cosh (1+j)mx}{\cosh (1+j)ma}\right) \qquad . \qquad . \ (363\text{A})$$

Now $\quad \cosh (1+j)\theta = \cosh \theta \cos \theta + j \sinh \theta \sin \theta$

and on rationalizing we get

$$\sqrt{[\tfrac{1}{2}(\cosh 2\theta + \cos 2\theta)]} \,\underline{/\phi}$$

where $\tan \phi = \tanh \theta \tan \theta$.

Substituting these results in equation (363A), we get

$$H_x = H_0\left(\frac{\cosh 2mx + \cos 2mx}{\cosh 2ma + \cos 2ma}\right)^{\frac{1}{2}} \underline{/\phi_1 - \phi_2} \qquad . \quad (364)$$

where
$$\tan \phi_1 = \tanh mx \tan mx \qquad . \qquad . \ (365\text{A})$$
$$\tan \phi_2 = \tanh ma \tan ma \qquad . \qquad . \ (365\text{B})$$

When the external magnetic force is given by $H_0 \cos \omega t$, then the value of H is

$$H_x = H_0\left[\frac{\cosh 2mx + \cos 2mx}{\cosh 2ma + \cos 2ma}\right]^{\frac{1}{2}} \cos (\omega t - \gamma_x) \ . \quad (366)$$

where $\gamma_x = \phi_1 - \phi_2$.

At the centre line we get

$$H_c = H_0\left(\frac{2}{\cosh 2ma + \cos 2ma}\right)^{\frac{1}{2}} \cos (\omega t - \gamma_c) \quad . \quad (367)$$

i.e. along the centre line, H has its minimum value.

The mean value of H is given by

$$H_m = \frac{1}{a}\int_0^a H_x dx$$

$$\therefore \qquad H_m = \frac{H_0}{ma\sqrt{2}}\left(\frac{\cosh 2ma - \cos 2ma}{\cosh 2ma + \cos 2ma}\right)^{\frac{1}{2}} \qquad . \quad (368)$$

SPACE PHASE ANGLE. The angle γ_x will represent the displacement of H as x is varied from a to 0, or as we move from the outside face to the centre. Now

$$\tan \gamma = \tan (\phi_1 - \phi_2)$$

and by expansion we get

$$\tan \gamma = \frac{\sinh m(a-x) \sin m(a+x) + \sinh m(a+x) \sin m(a-x)}{\cosh m(a-x) \cos m(a+x) + \cosh m(a+x) \cos m(a-x)} \quad . \quad . \quad . \quad (369)$$

At $x = a$ we have $\tan (\phi_1 - \phi_2) = 0$, i.e. the angle of lag is zero.

At $x = 0$, then

$$\tan \gamma_c = \tanh ma \tan ma = \tan \phi_2$$

Now, when $(\phi_1 - \phi_2) = 0$, then H_x must be in phase with H_0.

From equation (369) we get

$$mx = ma - 2s\pi^*$$

or

$$x = a - \frac{s}{m} . 2\pi$$

At depths given by $2\pi\frac{s}{m}$ from the surface, or at depths of $2\pi/m$, $4\pi/m$, . . . the magnetic force is in phase with that at the face.

When $(\phi_1 + \phi_2) = 0$, we have that at depths π/m, $3\pi/m$, . . . the forces are in phase opposition. The depth $2\pi/m$ is defined as the wavelength, or

$$\lambda = \frac{2\pi}{\sqrt{(2\pi\mu\omega/\rho)}} = \sqrt{(\rho/\mu f)} \text{ in } ab\text{-units}$$

or in practical units, with $m^2 = \dfrac{2\pi\mu\omega}{\rho} \times 10^{-9}$, we get

$$\lambda = 10^4\sqrt{\left(\frac{10\rho}{\mu f}\right)} \text{ cm.} \quad . \quad . \quad . \quad (370)$$

EFFECTIVE PENETRATION. This is defined by Ewing as the thickness of a surface layer which, at a constant flux density

* s = an integer.

B_0, would give the same flux as exists in the lamination. Let the thickness be d. Then

$$B_0 d = \int_0^a B_x dx$$

On integrating we get, for large values of ma, that

$$d = \frac{1}{(1+j)m}$$

Or in practical units

$$d = 1/m\sqrt{2} = 3\,570\sqrt{(\rho/\mu f)} \text{ cm.} \qquad . \quad (371)$$

Thus with $\mu = 1\,000$ and $\rho = 10^{-5}$, at 1 000 c.p.s. the depth is 0·0113 cm., and at 10^5 c.p.s. is 0·00113 cm.

4. **Eddy Current Density and Iron Losses.** From equation (357) we have that

$$\sigma = -\frac{1}{4\pi}\frac{\partial H}{\partial x}$$

On substituting for H and differentiating, we get

$$\sigma = -\frac{H_0}{4\pi}\left[(1+j)m\frac{\sinh\,(1+j)mx}{\cosh\,(1+j)ma}\right]$$

or

$$\sigma = A\sinh\,(1+j)mx \quad . \quad . \quad . \quad (372)$$

where $A = -\dfrac{H_0}{4\pi}(1+j)m\dfrac{1}{\cosh\,(1+j)ma}$.

The loss in a plate may be formulated as $(\rho\int_0^a \sigma^2 dx)$ per a cm.³

$$\text{The mean value of } \sigma^2 = \frac{|A|^2}{2a}\int_0^a(\cosh 2mx - \cos 2mx)dx$$

$$= \frac{|A|^2}{4ma}(\sinh 2ma - \sin 2ma)$$

As the r.m.s. value of a harmonic quantity is $(1/\sqrt{2})$ times the maximum, so, over a complete period, the effective value of σ^2 is

$$\frac{|A^2|}{8ma}(\sinh 2ma - \sin 2ma)$$

and the loss becomes

$$\frac{|A^2|\rho}{8ma}(\sinh 2ma - \sin 2ma) \times 10^{-7} \text{ watt} \qquad . \quad (373)$$

Substituting for $|A^2|$, the mean watt loss per cm.^{3} is

$$W_e = \frac{H_0^2 \rho m}{32\pi^2 a}\left(\frac{\sinh 2ma - \sin 2ma}{\cosh 2ma + \cos 2ma}\right) \times 10^{-7} \text{ watt}$$

In terms of mean density, we get

$$W_e = B_m^2 \cdot \frac{\omega^2 a}{m\rho}\left(\frac{\sinh 2ma - \sin 2ma}{\cosh 2ma - \cos 2ma}\right) \times 10^{-7} \text{ watt} \quad . \quad (374)$$

For large values of ma this expression becomes

$$W_e = \frac{H_0^2}{16\pi a}\sqrt{(\rho\mu f)} \times 10^{-7} \text{ watt} \quad . \quad . \quad (375)$$

5. **Phase Angle between Current and E.M.F. in an Exciting Coil of a Closed Magnetic Circuit.** The power input to such a circuit is $\frac{1}{2}V_{max}I_{max}\cos\phi$, where V_{max} and I_{max} are the maximum voltage and current. This expression may be written as

$$\tfrac{1}{2}\text{ Volts/Turn} \times \text{Amp. Turns} \cdot \cos\phi$$

So that the loss per cm.^{3} is given in r.m.s. values as

$$\frac{V_t \times \widetilde{A}/\text{cm.}}{\text{area}} \cos\phi$$

Now, volts per turn $= \omega B_m \times$ area and $4\pi\mu\widetilde{A}/\text{cm.} =$ flux density at face (B_0)

$$= (\sqrt{2})maB_m\sqrt{\left(\frac{\cosh 2ma + \cos 2ma}{\cosh 2ma - \cos 2ma}\right)}$$

The input power is

$$\frac{\omega B_m^2}{4\pi\mu}(\sqrt{2})ma\sqrt{\left(\frac{\cosh 2ma + \cos 2ma}{\cosh 2ma - \cos 2ma}\right)} \cdot \cos\phi \quad . \quad (376)$$

which is equal to iron losses given in equation (374).

$$\text{Hence} \qquad \cos\phi = \frac{\dfrac{\omega^2 a}{m\rho} \cdot 4\pi\mu \cdot \left(\dfrac{\sinh 2ma - \sin 2ma}{\cosh 2ma - \cos 2ma}\right)}{(\sqrt{2})\omega ma\sqrt{\left(\dfrac{\cosh 2ma + \cos 2ma}{\cosh 2ma - \cos 2ma}\right)}}$$

$$= \frac{1}{\sqrt{2}}\frac{\sinh 2ma - \sin 2ma}{\sqrt{(\cosh^2 2ma - \cos^2 2ma)}} \quad . \quad (377)$$

In the limit, when $2ma$ is large, then

$$\cos\phi = 1/\sqrt{2}$$

6. **Effect of Hysteretic Angle.** The effect of hysteresis has been investigated by various authors. As a full development would require too much space, we give a summary of the results given by Latour.*

Denoting the hysteresis angle by τ, $\sqrt{(1 + \sin\tau)}$ by α and $\sqrt{(1 - \sin\tau)}$ by β, we have the following relation—

$$B_{max} = (A\rho m/\omega)\sqrt{(\cosh 2m\alpha x + \cos 2m\beta x)} \quad . \quad . \quad (378)$$

$$\text{where } A = \frac{\omega a B_m\sqrt{2}}{\rho\sqrt{(\cosh 2m\alpha x - \cos 2m\beta a)}}$$

$$= \frac{\omega}{\rho m}\,\frac{4\pi\mu I}{\sqrt{(\cosh 2m\alpha a + \cos 2m\beta a)}} \quad . \quad (379)$$

$$\mu_{app} = \frac{\mu}{ma\sqrt{2}}\left(\frac{\cosh 2m\alpha a - \cos 2m\beta a}{\cosh 2m\alpha a + \cos 2m\beta a}\right)^{\frac{1}{2}} \quad . \quad . \quad (380)$$

$$W_e = \frac{\omega^2 a}{4m\rho}\left(\frac{\dfrac{\sinh 2m\alpha a}{\alpha} - \dfrac{\sin 2m\beta a}{\beta}}{\cosh 2m\alpha a - \cos 2m\beta a}\right)B_m^{\ 2} \quad . \quad . \quad (381)$$

$$W_h = \frac{\omega^2 a \sin\tau}{4m\rho}\left(\frac{\dfrac{\sinh 2m\alpha a}{\alpha} + \dfrac{\sin 2m\beta a}{\beta}}{\cosh 2m\alpha a - \cos 2m\beta a}\right)B_m^{\ 2} \quad . \quad (382)$$

$$W_e + W_h = \frac{\omega^2 a}{4m\rho}\left(\frac{\alpha\sinh 2m\alpha a - \beta\sin 2m\beta a}{\cosh 2m\alpha a - \cos 2m\beta a}\right)^{\frac{1}{2}} B_m^{\ 2} \quad . \quad (383)$$

$$\cos\phi = \frac{(1/\sqrt{2})(\alpha\sinh 2m\alpha a - \beta\sin 2m\beta a)}{(\cosh^2 2m\alpha a - \cos^2 2m\beta a)^{\frac{1}{2}}} \quad . \quad . \quad (384)$$

On putting $\alpha = \beta = 1$ in the above expressions, we obtain the previous results, which neglect hysteresis.

EXPERIMENTAL DATA. The following test results are taken from Dannatt.†

Material	Cos ϕ	
	50 c.p.s.	H.F.
Silicon steel 0·32 mm.	0·58	0·78
Silicon steel 0·052 mm.	0·18	0·72
Mu metal 0·25 mm.	0·8	0·86
Rho metal 0·35 mm.	0·6	0·84

* LATOUR: *Electrician* (1919), p. 219.

† DANNATT: "Magnetic Properties of Ferro-magnetic Laminae," *Journ. I.E.E.* (1936), Vol. 79, p. 667.

Thus at high frequencies the power factors are greater than the limiting value of 0·707. From the same paper the following values of permeability are taken—

Material	Permeability	
	50 c.p.s.	H.F.
Silicon steel 0·32 mm.	880	230
Silicon steel 0·052 mm.	480	210
Mu metal 0·25 mm.	9 500	180
Rho metal 0·35 mm.	1 100	150

Again the theory does not account for all the variation in permeability, so that it would appear that a complete theory must take in account the effect of the surface layer, the size of the iron crystals, and the direction of magnetization with respect to the crystal.

7. Eddy Currents in Armature Bars. We will assume that the slot leakage lines go straight across the slot and that the iron is of infinite permeability. Under these conditions the value of the leakage flux is directly proportional to the width of the slot, and, due to the pulsation of these lines, an e.m.f. will be induced in the conductor. The presence of this e.m.f. causes distortion of the current distribution within the conductor. However, all leakage lines do not cause distortion. Certain of the flux will be linked with the whole conductor and will induce the same e.m.f. in all sections. The effective flux will be that due to currents below the section in question. For example, in Fig. 38 the line shown as 1 is effective, whereas that shown as 2 is ineffective so far as conductor B is concerned. The total disturbing flux in the shaded element is

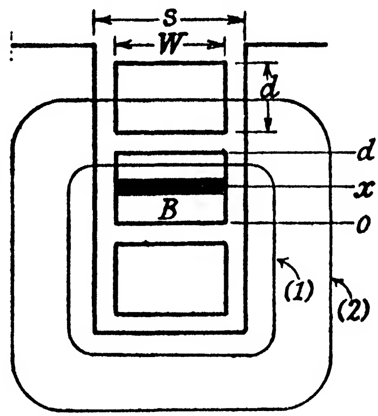

FIG. 38

$$\phi = \frac{4\pi I_0}{s}(d-x)l + \int_x^d 4\pi \int_0^x (W\sigma dx)\frac{l dx}{s} \quad . \quad (385)$$

where I_0 is the current in ab-amperes in all conductors below the conductor considered and whose flux passes within the region x to d of the conductor B,

σ is the current density in ab-amperes,

$\int_0^x W\sigma dx$ is the current in the conductor B below the section, i.e. in the region O to x,

l is the armature length.

The voltage across 1 cm. length of the conductor is

$$V = ri + \frac{d\phi}{dt}$$

when i is the current through the element.

As this voltage must be the same for all elements of the conductor, we have

$$\frac{dV}{dx} \equiv 0 \quad \text{or} \quad r\frac{di}{dx} + \frac{d}{dx} \cdot \frac{d\phi}{dt} = 0$$

But $ri = \rho l\sigma$; so, on substituting for i and ϕ, we get

$$\rho l \frac{d\sigma}{dx} - 4\pi \frac{l}{s}\frac{dI_0}{dt} - \frac{4\pi l}{s}\int_0^x W \cdot \frac{d\sigma}{dt} \cdot dx = 0 \quad . \quad (386)$$

With currents varying harmonically, we get

$$\frac{d\sigma}{dx} - j\frac{4\pi\omega}{\rho s} \cdot I_0 - j\frac{4\pi\omega}{\rho s}\int_0^x W\sigma dx = 0 \quad . \quad (387)$$

Further differentiation gives

$$\frac{d^2\sigma}{dx^2} - j\frac{4\pi\omega}{\rho s} W\sigma = 0$$

or

$$\frac{d^2\sigma}{dx^2} = \alpha^2\sigma$$

$$\text{where } \alpha^2 = j\frac{4\pi\omega}{\rho s} W \text{ or } \alpha = (1 + j)\sqrt{\left(\frac{2\pi\omega}{\rho s} W\right)}.$$

The solution of this equation is

$$\sigma_x = A \cosh \alpha x + B \sinh \alpha x \quad . \quad . \quad (388)$$

Substitution of (387) in (388) gives

$$B = \alpha I_0/W$$

If I_1 be the current in the conductor, then

$$I_1 = \int_0^d W\sigma dx$$

Hence $\qquad A = \alpha[(I_1/\sinh \alpha d) - I_0 \tanh \tfrac{1}{2}\alpha d]/W$

So we obtain

$$\sigma_x = \alpha d\left[\sigma_1 \frac{\cosh \alpha x}{\sinh \alpha d} + \sigma_0 \sinh \alpha x - \sigma_0 \tanh \tfrac{1}{2}\alpha d \cosh \alpha x\right] \quad . \quad (389)$$

For the slot arrangement of Fig. 39 the current densities are given in the following table, where $\sigma_0 = \sigma_1 = \sigma_n$.

Level	Density
1	$\sigma_n\alpha d\,[\coth \alpha d + \tanh \tfrac{1}{2}\alpha d]$
2	$\sigma_n\alpha d\,[\tfrac{1}{2}\,\mathrm{coshec}\,\tfrac{1}{2}\alpha d]$
3	$\sigma_n\alpha d\,[\mathrm{coshec}\,\alpha d - \tanh \tfrac{1}{2}\alpha d]$
4	$\sigma_n\alpha d\,[\coth \alpha d]$
5	$\sigma_n\alpha d\,[\tfrac{1}{2}\,\mathrm{coshec}\,\tfrac{1}{2}\alpha d]$
6	$\sigma_n\alpha d\,[\mathrm{coshec}\,\alpha d]$

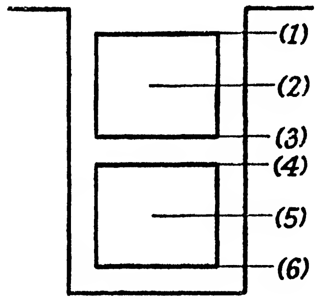

FIG. 39

8. **Voltage Drop in a Conductor per Centimetre Length.** The total voltage drop in a conductor is due to both sets of fluxes. The "effective" flux will set up a voltage, which is the same in all sections, so we use the section where $x = d$ and get

$$V = \rho\sigma_d = \frac{\rho\alpha}{W}(I_1 \coth \alpha d + I_0 \tanh \tfrac{1}{2}\alpha d) \text{ per cm.} \quad . \quad (390)$$

The "ineffective" flux is due to the mutual action of currents in all conductors lying below the one under consideration plus the self-induction due to the current in the conductor itself, or

$$\phi = \frac{4\pi d}{s} I_b + \int_0^d \frac{4\pi}{s} \int_0^x (W\sigma dx) dx$$

$$= \frac{4\pi}{\alpha s}\left\{I_1 \frac{\cosh \alpha d - 1}{\sinh \alpha d}\right.$$

$$\left. - I_0 [\tanh \tfrac{1}{2}\alpha d(\cosh \alpha d - 1) - \sinh \alpha d] + \alpha d(I_b - I_0)\right\}$$

$$= \frac{\rho}{j\omega} \cdot \frac{1}{Wd} [(\tfrac{1}{2}I_1 + I_0)2\alpha d \tanh \tfrac{1}{2}\alpha d + \alpha^2 d^2 (I_b - I_0)]$$

So the induced voltage is given by

$$e = \frac{\rho}{Wd}[(\tfrac{1}{2}I_1 + I_0)2\alpha d \tanh \tfrac{1}{2}\alpha d + \alpha^2 d^2(I_b - I_0)] \quad . \quad . \quad (391)$$

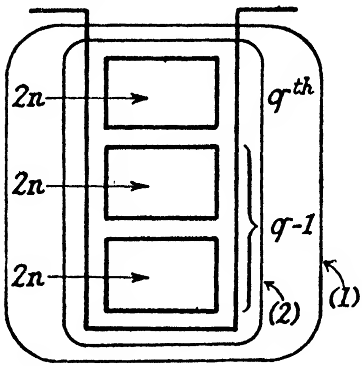

FIG. 40

A proper combination of V and e gives the voltage drop in the conductor.

For a full-pitch winding with solid conductors (Fig. 40), and if of the $2n$ conductors in each layer we consider the qth conductor from the bottom of the slot, then, since there are $(q-1)$ conductors below the conductor, the effective current is $I_0 = I_1(q-1) = I_b$. So that we get

$$\left.\begin{aligned} V &= \frac{\rho\alpha l}{W}[I_1 \coth \alpha d + I_1(q-1)\tanh \tfrac{1}{2}\alpha d] \\ (q-1)e &= \frac{\rho I_1 l}{Wd}[(\tfrac{1}{2} + q - 1)2\alpha d \tanh \tfrac{1}{2}\alpha d](q-1) \end{aligned}\right\} . \quad (392)$$

and $\dfrac{\rho}{Wd} = \dfrac{R}{2nl}$.

So we obtain the total impedance as,

$$Z = \frac{R}{2n}\sum_{1}^{2n}\left(M + \frac{q-1}{2}N\right) + \frac{R}{2n}\sum_{1}^{2n}(q-1)(q-\tfrac{1}{2})N \quad (393)$$

$$= R\left(M + \frac{4n^2-1}{3}N\right)$$

where $M = \alpha d \coth \alpha d = M_r + jM_x$,

$N = 2\alpha d \tanh \tfrac{1}{2}\alpha d = N_r + jN_x$.

So we have that

$$R = \frac{\rho}{Wd}\left[M_r + \frac{4n^2-1}{3}N_r\right] \quad . \quad . \quad (394)$$

$$X = \frac{\rho}{Wd}\left[M_x + \frac{4n^2-1}{3}N_x\right] \quad . \quad . \quad (395)$$

For further development, reference should be mae to the papers listed in the Bibliography.

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CHAPTER X

FOURIER'S SERIES AND INTEGRALS

HEAVISIDE frequently points out the parallel between his methods and those of Fourier. We now use the latter to explain some points of the expansion theorem. As the mathematical proofs can be found in standard mathematical texts, we will merely sketch the theory. Solutions of differential equations, where the disturbing force is expressed as a series, are dealt with and exhibit no novel features.

The integral is a natural development of the series—in electrical language it gives the frequency spectrum of the array of impulses denoted by the series. When the impressed force is given by the integral the solution is readily obtained, provided that the resultant integral can be evaluated. An explanation of the unit function can be derived from this theory, and the function is seen to consist of terms of all frequencies. A very interesting use of the integral is in an integral equation, which in one form is known as Carson's equation, and in another as Bromwich's equation. From these equations the equivalencies of a great many operators have been evaluated by Campbell (see Bibliography).

1. **General Theory of Periodic Functions.** If $f(t)$ is a single-valued function, then in the interval $-s < t < s$ it may be represented by

$$\left.\begin{aligned} f(t) = \tfrac{1}{2}A_0 + A_1 \cos\frac{\pi t}{s} + A_2 \cos\frac{2\pi t}{s} + \ldots \\ + B_1 \sin\frac{\pi t}{s} + B_2 \sin\frac{2\pi t}{s} + \ldots \end{aligned}\right\} \qquad (396)$$

With the proviso that $f(t)$ has

(i) a limited number of discontinuities,

(ii) a finite number of maxima and minima, and

(iii) that $\int_{-s}^{s} [f(t)dt]^2$ exists.

We have $$A_n = \frac{1}{s}\int_{-s}^{s} f(\lambda)\cos\frac{n\pi\lambda}{s}\,d\lambda \qquad (397)$$

and $$B_n = \frac{1}{s}\int_{-s}^{s} f(\lambda)\sin\frac{n\pi\lambda}{s}\,d\lambda \qquad (398)$$

The series may be written in several forms, for example with $\pi/s = \omega$

(i) If $f(t)$ is an even function, i.e. $f(t) = +f(-t)$, then

$$f(t) = \frac{1}{s}\int_0^s f(\lambda)d\lambda + \frac{1}{\pi}\sum_1^\infty \cos n\omega t\int_{-2\pi}^{2\pi} f(\lambda)\cos n\omega\lambda d(\omega\lambda) \quad (399)$$

(ii) If $f(t)$ is an odd function, i.e. $f(t) = -f(-t)$, then

$$f(t) = \frac{1}{\pi}\sum_1^\infty \sin n\omega t\int_{-2\pi}^{2\pi} f(\lambda)\sin n\omega\lambda \,.\, d(\omega\lambda) \quad . \quad . \quad (400)$$

There are other forms of these expressions, but all converge to $f(t)$ in the interval $-s < t < s$, and at points of discontinuity become equal to

$$\underset{u\to 0}{Lt}\ \tfrac{1}{2}[f(t+u) - f(t-u)]$$

2. **Determination of the Wave Form of a Function.** This is the operation inverse to that with which engineers are familiar. Here the series is given, and it is required to find the wave shape. If

$$f(t) = \frac{2}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi}{s}t \quad . \quad . \quad . \quad . \quad . \quad . \quad (401)$$

or operationally

$$f(t) = \frac{2}{\pi}\sum_1^\infty \frac{1}{n}\frac{\frac{n\pi}{s}p}{p^2 + \left(\frac{n\pi}{s}\right)^2} = \frac{1}{sp}\sum_1^\infty \frac{2}{1 + \left(\frac{n\pi}{sp}\right)^2},$$

then on expansion

$$f(t) = \frac{1}{sp}\left[\frac{2}{1+\left(\frac{\pi}{sp}\right)^2} + \frac{2}{1+\left(\frac{2\pi}{sp}\right)^2} + \cdots\right] \quad . \quad (402)$$

But we have the following partial fraction development from the theory of complex functions—

$$sp\coth sp = 1 + \frac{2}{1+\left(\frac{\pi}{sp}\right)^2} + \frac{2}{1+\left(\frac{2\pi}{sp}\right)^2}\cdots \quad . \quad (403)$$

Equating (402) and (403) we get

$$f(t) = \frac{1}{sp}(sp \coth sp - 1) \quad . \quad . \quad . \quad . \quad (404)$$

Integrating the last term, and expanding the coth term, we have

$$f(t) = 1 - \frac{t}{s} + 2\varepsilon^{-2sp} + 2\varepsilon^{-4sp} \ldots \quad . \quad . \quad . \quad (405)$$

Taylor's theorem gives

$$\varepsilon^{-2sp} f(t) = f(t - 2s)$$

So, in equation (405), when $t < 2s$, all exponential terms are zero, and the value of $f(t)$ is $(1 - t/s)$, i.e. a straight line. For the period $t = 2s$ to $t = 4s$ the first exponential term is present, and we get

$$f(t) = [1 - t/s + 2f(t - 2s)] = [3 - t/s]\, t > 2s$$

We are now able to draw out the wave shape as shown in Fig. 41.

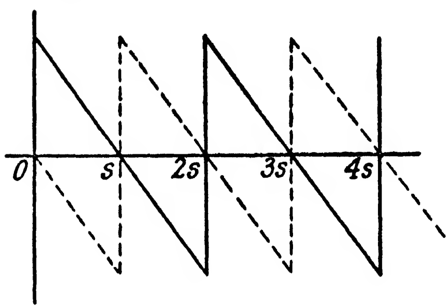

FIG. 41

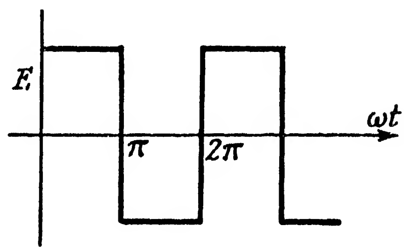

FIG. 42

If now we subtract a wave displaced by 180° on the former, we get the telegraph dot wave. Mathematically, we have

$$E = \frac{E}{\pi}[\sin \omega t + \tfrac{1}{2} \sin 2\omega t \ . \ . \ .] -$$

$$- \frac{E}{\pi}[\sin (\omega t + \pi) + \tfrac{1}{2} \sin 2(\omega t + \pi) \ . \ . \ .]$$

$$= \frac{2E}{\pi}[\sin \omega t + \tfrac{1}{3} \sin 3\omega t + . \ . \ .]$$

$$= \frac{2E}{\pi} \sum \frac{1}{n} \sin n\omega t \qquad (n \text{ odd}) \text{ (Fig. 42)} \quad . \quad . \quad (406)$$

Now, on displacing the time axis downward, we get a wave formulated by

$$e(t) = \frac{E}{2} + \frac{2E}{\pi}\sum \frac{\sin n\omega t}{n} \quad \text{(Fig. 43)} \qquad . \quad (407)$$

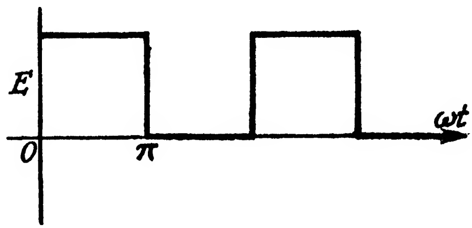

FIG. 43

3. **Application of Fourier's Series to Circuits.** CASE 1. Let a voltage represented by $E = \frac{2}{\pi} E_0 \sum_1^\infty \frac{\sin n\omega t}{n}$ be applied to an RL circuit. Then we get

$$i = \frac{2}{\pi} E_0 \sum_1^\infty \frac{\sin n\omega t}{n(R + Lp)} \cdot (1)$$

On rationalizing and discarding imaginaries, we get

$$i = \frac{2}{\pi} E_0 \sum \frac{1}{n}\left[\frac{R\cos n\omega t + n\omega L\sin n\omega t}{R^2 + n^2\omega^2L^2} - \frac{\varepsilon^{-(R/L)t}}{R^2 + n^2\omega^2L^2}\right]$$

$$= \frac{2}{\pi} E_0 \sum \frac{1}{nZ_n}[\cos(n\omega t - \phi_n) - \cos\phi_n\,\varepsilon^{-(R/L)t}] \qquad . \quad (408)$$

where $\tan\phi_n = \frac{n\omega L}{R}$ and $Z_n = \sqrt{(R^2 + n^2\omega^2L^2)}$.

CASE 2. With a voltage $E = E_0 + \sum_1^\infty E_n \cos(n\omega t + \xi_n)$ applied to an RLC circuit, we have

$$[LCp^2 + RCp + 1]q = C[E_0 + \sum_1^\infty E_n \cos(n\omega t + \xi_n)]$$

Or, on substituting $\alpha = R/2L$ and $\omega_0^2 = 1/LC$, we get

$$(p^2 + 2\alpha p + \omega_0^2)q = [E_0 + \sum_1^\infty E_n \cos(n\omega t + \xi_n)]/L$$

With oscillatory conditions, the first term is

$$Lq_1 = E_0\left[\frac{1}{\omega_0^2} + \frac{\varepsilon^{-\alpha t}}{2j\beta}\left(\frac{\varepsilon^{j\beta t}}{-\alpha + j\beta} - \frac{\varepsilon^{-j\beta t}}{-\alpha - j\beta}\right)\right] \quad (409)$$

This yields, after some simplification

$$Lq_1 = E_0\left[\frac{1}{\omega_0^2} - \frac{\varepsilon^{-\alpha t}\cos(\beta t - \phi)}{\beta\sqrt{(\alpha^2 + \beta^2)}}\right] \quad . \quad . \quad (410)$$

where $\beta = \sqrt{(\omega_0^2 - \alpha^2)}$ and $\tan\phi = \alpha/\beta$.

For the second term, omitting the summation sign, we get after rationalizing and discarding imaginaries

$$Lq_2 = E_n\left\{\frac{\cos(n\omega t + \xi_n - \phi_n)}{\sqrt{[(2\alpha n\omega)^2 + (\omega_0^2 - n^2\omega^2)^2]}} + \frac{\varepsilon^{-\alpha t}}{\beta}\left[\frac{\alpha\sin(\xi_n + \beta)t - (n\omega - \beta)\cos(\xi_n + \beta)t}{\alpha^2 + (n\omega - \beta)^2} - \frac{\alpha\sin(\xi_n - \beta)t - (n\omega + \beta)\cos(\xi_n - \beta)t}{\alpha^2 + (n\omega + \beta)^2}\right]\right\} \quad . \quad (411)$$

The full expression is given by

$$q = \frac{E_0}{L}\left[\frac{1}{\omega_0^2} - \frac{\varepsilon^{-\alpha t}}{\omega_0\beta}\cos(\beta t - \phi)\right] + \sum_{n=1}^{n=\infty}\frac{E_n}{L}\left\{\frac{\cos(n\omega t + \xi_n - \phi_n)}{\sqrt{[(2\alpha n\omega)^2 + (\omega_0^2 - \omega^2 n^2)^2]}} - \frac{\varepsilon^{-\alpha t}}{\beta_n}\left[\frac{\cos(p_n t + \xi_n + \theta_n')}{\sqrt{[\alpha^2 + (n\omega - \beta_n)]^2}} - \frac{\cos(\beta_n t - \xi_n + \theta_n'')}{\sqrt{[\alpha^2 + (n\omega + \beta)^2]}}\right]\right\} \quad (412)$$

where $\tan\phi_n = \dfrac{\alpha n\omega}{\omega_0^2 - n^2\omega^2}$, $\tan\theta_n' = \dfrac{\alpha}{n\omega - \beta}$, and $\tan\theta_n'' = \dfrac{\alpha}{n\omega + \beta}$.

The steady-state term is given by

$$Q = \frac{E_0}{L}\frac{1}{\omega_0^2} + \sum_1^\infty \frac{E_n}{L}\frac{\cos(n\omega t + \xi_n - \phi_n)}{\sqrt{[(2\alpha n\omega)^2 + (\omega_0^2 - n^2\omega^2)^2]}} \quad . \quad (413)$$

and on differentiation we get the steady-state current as

$$I = \sum_1^\infty \frac{n\omega E_n}{L}\frac{\sin(n\omega t + \xi_n - \phi_n)}{\sqrt{[(2\alpha n\omega)^2 + (\omega_0^2 - n^2\omega^2)^2]}} \quad . \quad . \quad (414)$$

CASE 3. RESPONSE OF AN OSCILLOGRAPH TO A DAMPED SINE WAVE. The force equation for an oscillograph string is

$$(p^2 + \alpha p + \omega_0{}^2)y = F_0 + \sum_1^\infty F_n \varepsilon^{-\mu_n t} \cos n\omega t \,. \quad (1)$$

The solution of this equation is deduced readily from the preceding as

$$\begin{aligned} y = F_0 &\left[\frac{1}{\omega_0{}^2} - \frac{\varepsilon^{-\alpha t}}{\omega_0 \beta} \cos(\beta t - \phi)\right] \\ &+ \sum_1^\infty \frac{F_n \,\varepsilon^{-\mu_n t} \cos(n\omega t - \phi_n)}{\sqrt{\{[\omega_0{}^2 - n^2\omega^2 + \mu(\mu - \alpha)]^2 + n^2\omega^2(2\mu - \alpha)^2\}}} \quad (415) \\ &\quad + \text{two other transient terms} \end{aligned}$$

where $\tan \phi_n = \dfrac{n\omega(2\mu - \alpha)}{\omega_0{}^2 - n^2\omega^2 + \mu(\mu - \alpha)}$.

4. **Fourier's Integral.** The series may be written as

$$\begin{aligned} f(t) = \frac{1}{2s}\int_0^s f(\lambda)d\lambda &+ \sum_1^\infty \frac{1}{s}\cos\frac{n\pi t}{s}\int_{-s}^s f(\lambda)\cos\left(\frac{n\pi\lambda}{s}\right)d\lambda \\ &+ \sum_1^\infty \frac{1}{s}\sin\frac{n\pi t}{s}\int_{-s}^s f(\lambda)\sin\left(\frac{n\pi\lambda}{s}\right)d\lambda \end{aligned}$$

or, discarding the first term

$$\begin{aligned} f(t) = \sum_1^\infty \frac{1}{s}\int_{-s}^s f(\lambda)\Bigg[&\cos\left(\frac{n\pi t}{s}\right)\cos\left(\frac{n\pi\lambda}{s}\right) \\ &+ \sin\left(\frac{n\pi t}{s}\right)\sin\left(\frac{n\pi\lambda}{s}\right)\Bigg]d\lambda \quad . \quad (416) \end{aligned}$$

As $\cos(-\theta) = \cos\theta$, we get

$$f(t) = \tfrac{1}{2}\sum_{-\infty}^\infty \frac{1}{s}\int_{-s}^s f(\lambda)\cos\frac{n\pi}{s}(t - \lambda)d\lambda \quad . \quad . \quad . \quad (417)$$

Now let the period become infinitely large. With $m = n\pi/s$ the step π/s will become dm in the limit, or

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty dm \int_{-\infty}^\infty f(\lambda)\cos m(t - \lambda)d\lambda \quad . \quad . \quad . \quad (418)$$

Putting $m = \omega$, and noting $\cos \omega t = \frac{1}{2}(\varepsilon^{j\omega t} + \varepsilon^{-j\omega t})$, we get

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(\lambda)\varepsilon^{-j\omega(t-\lambda)}\, d\lambda \quad . \quad . \quad . \quad . \quad (419)$$

These integrals are subject to the same limitations as the series. For an even function we get

$$f(t) = \frac{2}{\pi}\int_0^{\infty} \cos mt\, dm \int_0^{\infty} f(\lambda) \cos m\lambda \,.\, dm . \quad . \quad . \quad (420)$$

If we write $$g(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos mt\, f(m)\, dm \quad . \quad (421)$$

then

$$f(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos mt\, g(m)\, dm \quad . \quad . \quad . \quad . \quad (422)$$

These formulae are known as the Fourier cosine transforms. The sine transforms are given by

$$h(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \sin mt\, f(m)\, dm \quad . \quad . \quad . \quad (423)$$

$$f(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \sin mt\, h(m)\, dm \quad . \quad . \quad . \quad (424)$$

5. **Conversion of the Series to the Integral.** As an example of the conversion we consider the series

$$e(t) = \sum_1^{\infty} \frac{2E_0}{\pi}\frac{\sin n(\pi/s)t}{n} \quad (n \text{ odd}) \quad . \quad . \quad (425)$$

Now let the period become infinitely large. By the method of last section we get $m = n\pi/s = \omega$ and $dm = \pi/s = d\omega$.

$$\therefore \frac{2E_0}{\pi n} = \frac{2E_0}{\pi} \cdot \frac{\pi}{ms} = \frac{2E_0}{\pi} \cdot \frac{dm}{m} = \frac{2E_0}{\pi}\frac{d\omega}{\omega}$$

So $$e(t) = 2\int_0^{\infty} \frac{E_0}{\pi}\frac{\sin \omega t}{\omega}\, d\omega \quad . \quad . \quad . \quad (426)$$

On plotting the amplitude of the component harmonics against ω for equations (425) and (426), in the case of (425) we get a row of impulses at definite values of ω (Fig. 44*a*), while for (426) we get a smooth graph or continuous frequency spectrum (Fig. 44*b*). The contribution or response made by frequencies between ω and $(\omega + d\omega)$ is given by the shaded area.

For a square wave given by

$$E = E_0\left[\tfrac{1}{2} + \frac{1}{\pi}\sum_{1}^{\infty}\frac{\sin n\left(\frac{\pi}{s}\right)t}{n}\right]$$

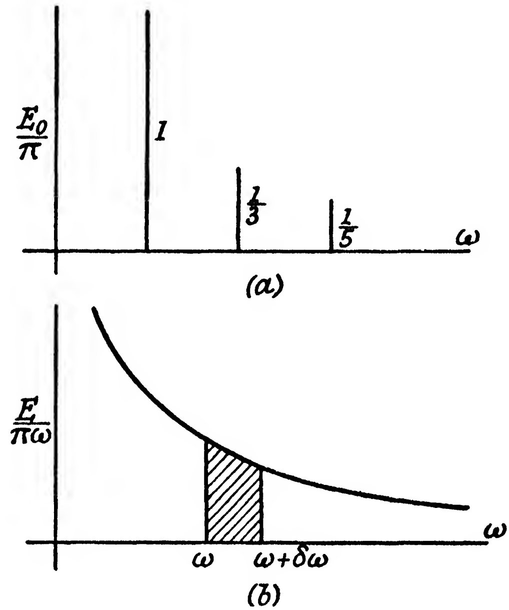

FIG. 44

when the period becomes infinitely large the wave is given by the integral

$$E = E_0\left[\tfrac{1}{2} + \frac{1}{\pi}\int_0^{\infty}\frac{\sin \omega t}{\omega}\,d\omega\right] \qquad . \qquad . \quad (427)$$

Carrying out the same process graphically, we get Fig. 45 (*a*). This is seen to correspond to the unit function of Chapter I.

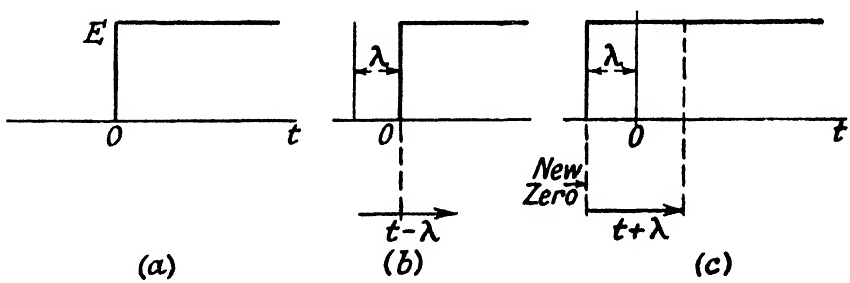

FIG. 45

So the admittance is either

$$A(t) = \frac{2}{\pi}\int_0^\infty \frac{R}{R^2 + X^2}\frac{\sin \omega t}{\omega}\, d\omega \quad . \quad . \quad . \quad (437)$$

or

$$= \frac{1}{R} - \frac{2}{\pi}\int_0^\infty \frac{X}{R^2 + X^2}\frac{\cos \omega t}{\omega}\, d\omega \quad . \quad . \quad (438)$$

(ii) For a condenser circuit we get

$$\frac{1}{Z} = \frac{R}{R^2 + 1/\omega^2 C^2} + j\frac{1/\omega C}{R^2 + 1/\omega^2 C^2}$$

So for unit voltage we get

$$A(t) = \frac{2}{\pi}\int_0^\infty \frac{R}{Z^2}\frac{\sin \omega t}{\omega}\, d\omega \quad . \quad . \quad . \quad . \quad (439)$$

The value of this integral is $\frac{\pi}{2R}\varepsilon^{-t/CR}$, so the current is

$$i = (E/R)\varepsilon^{-t/CR}$$

(iii) When the applied voltage is alternating we get

$$e = E \sin \omega_0 t\left[\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin \lambda t}{\lambda}\, d\lambda\right]$$

$$= \frac{E}{2}\sin \omega_0 t + \frac{E}{2\pi}\int_0^\infty \frac{1}{\lambda}[\cos(\omega_0 - \lambda)t - \cos(\omega_0 + \lambda)t]d\lambda \quad (440)$$

$$\text{or } e = E \cos \omega_0 t\left[\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin \lambda t}{\lambda}\, d\lambda\right]$$

$$= \frac{E}{2}\cos \omega_0 t + \frac{E}{2\pi}\int_0^\infty \frac{1}{\lambda}[\cos(\omega_0 - \lambda)t + \cos(\omega_0 + \lambda)t]\,.\,d\lambda \quad (441)$$

These equations are zero for negative values of t, and are $E \sin \omega t$ and $E \cos \omega t$ respectively for positive values. On integrating equation (440), it may be shown to be equivalent to

$$\frac{E}{2}\left[\sin \omega_0 t + \frac{1}{\pi}\int_0^\infty \frac{\omega_0}{\omega_0^2 - \lambda^2}\cos \lambda t\, d\lambda\right] \quad . \quad . \quad (442)$$

Now, suppose that this voltage is applied to a line $R = G = 0$, and that the far end of the line is terminated through a resistance R. Then

$$Z = Z_0\left[\frac{1 - \mu\varepsilon^{-2\alpha l}}{1 + \mu\varepsilon^{-2\alpha l}}\right]$$

and $$Y = \frac{1}{Z} = \frac{1}{Z_0}\left[\frac{1 + \mu\varepsilon^{-2\alpha l}}{1 - \mu\varepsilon^{-2\alpha l}}\right]^*$$
$$= Y_0[1 + 2\mu\varepsilon^{-2\alpha l} + 2\mu^2\varepsilon^{-4\alpha l}. \; . \; .] \; . \qquad . \quad (443)$$

Now, as $Z_0 = \sqrt{(L/C)}$, then $\alpha l = \sqrt{(LC)} \, . \, l = j\omega_0 T$, say, so we may write

$$Y = Y_0[1 + 2\mu\varepsilon^{-2j\omega_0 T} + 2\mu\varepsilon^{-4j\omega_0 T} \; . \; . \; .] = A(t) \quad (444)$$

When the applied volts are $E \sin \omega t$, the current is given by

$$i(t) = \frac{EY_0}{2}[1 + 2\mu\varepsilon^{-2j\omega_0 T} + \; . \; . \; .] \sin \omega_0 t$$
$$+ \frac{EY_0}{2\pi}\int_0^\infty (1 + 2\mu\varepsilon^{-2j\omega_0 T} \; . \; . \; .) \frac{\omega_0}{\omega_0^2 - \omega^2} \cos \omega t \, d\omega \quad (445)$$

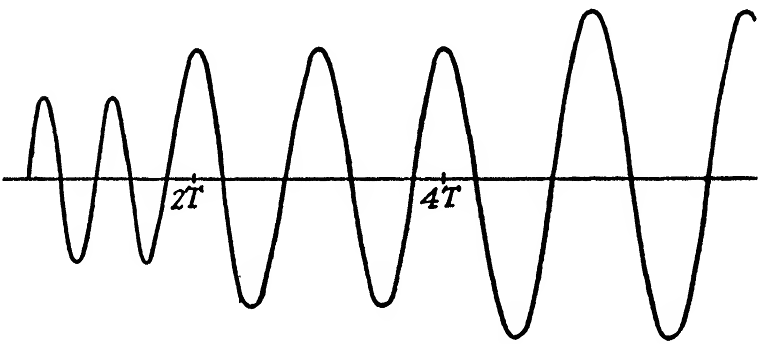

FIG. 46

Inspection of this equation reveals that the terms come in at different times; thus in the intervals

0 to $2T$ we get

$$\frac{EY_0}{2} \sin \omega_0 t + \frac{EY_0}{2\pi}\int_0^\infty \frac{\omega_0}{\omega_0^2 - \omega^2} \cos \omega t \, d\omega$$

$2T$ to $4T$ we get the above plus

$$\frac{EY_0}{2} \mu\varepsilon^{-2j\omega T_0} \sin \omega t + \frac{EY_0}{2\pi}\int_0^\infty 2\mu\varepsilon^{-2j\omega_0 T} \frac{\omega_0}{\omega_0^2 - \omega^2} \cos \omega t \, d\omega$$

and so on.

As $\varepsilon^{-2j\omega T_0} \cos \omega t = \cos (\omega t - 2\omega_0 T)$, it is seen that the increment is retarded in phase by $2\omega_0 T$ and is also attenuated by 2μ. The form of the wave with respect to time is shown in Fig. 46.

* See Section 3, Chapter VIII.

7. **The Carson Integral Equation.** Let us consider the differential equation

$$(a_0p^n + a_1p^{n-1} + \ldots a_n)y = f(t) \quad . \quad . \quad (446)$$

or

$$F(p) \,.\, y = f(t)$$

The formal solution of this equation is well known; but we follow a method due to van der Pol.* For the initial conditions let

$$p^n y = p^{n-1} y \ldots = y = 0$$

On multiplying both members of the equation by ε^{-pt} and integrating between 0 and ∞, we have

$$\int_0^\infty [a_0p^n + a_1p^{n-1} \ldots]y\varepsilon^{-pt}\,dt = \int_0^\infty f(t)\varepsilon^{-pt}\,dt \quad . \quad (447)$$

For the term $\int_0^\infty p^n y\, \varepsilon^{-pt}\,dt$, by partial integration, we get

$$[\varepsilon^{-pt} p^{n-1} y]_0^\infty + p\int_0^\infty (p^{n-1} y)\varepsilon^{-pt}\,dt \quad . \quad . \quad (448)$$

The first term of this expression is zero by terminal conditions, and so

$$\int_0^\infty (p^n y)\varepsilon^{-pt}\,dt = p\int_0^\infty (p^{n-1} y)\varepsilon^{-pt}\,dt$$

Repetition gives

$$\int_0^\infty (p^n y)\varepsilon^{-pt}\,dt = p^n\int_0^\infty y\varepsilon^{-pt}\,dt \quad . \quad . \quad (449)$$

So the first member of the equation becomes

$$[a_0p^n + a_1p^{n-1} + \ldots + a_n]\int_0^\infty y\varepsilon^{-pt}\,dt = H(p)\int_0^\infty y\varepsilon^{-pt}\,dt, \text{ say} \quad . \quad . \quad . \quad (450)$$

For the second member, we will let $f(t)$ represent the unit function, and so

$$\int_0^\infty f(t)\varepsilon^{-pt}\,dt = \frac{1}{p}$$

So we get

$$H(p)\int_0^\infty y\varepsilon^{-pt}\,dt = \frac{1}{p}$$

or

$$\frac{1}{p\,.\,H(p)} = \int_0^\infty y\varepsilon^{-pt}\,dt \quad . \quad . \quad (451)$$

* Bibliography.

This equation was first given by Carson, and by it we are able to interpret operational expressions in terms of the Fourier integral, as will be shown.

8. **Development of Operator Equivalencies.** We now summarize the methods used to develop the equivalencies of operators. In our work we have used several methods, and so we will refer the reader to these as examples.

(i) Expansion of operators by means of the binomial theorem, by Taylor's theorem, or by partial fraction development. Examples will be found in Chapter I.

(ii) Using known operators and deriving therefrom others by the use of the shifting theorem, Borel's theorem, etc. (see Chapter I.

(iii) Using the integral equation of the previous section. This is a particularly powerful method, as all known Fourier integrals can be given in operational form, and reference should be made to a list by Campbell (see Bibliography).

We have already used method ii to evaluate operators in the form of a product (Chapter I). The idea has been extended by Carson and by van der Pol,* to which we must refer the reader for further information.

Method iii has not been used so far in our work, and we now show how it is handled. From Section 7, we have the integral equation

$$\frac{1}{p \cdot H(p)} = \int_0^\infty y\varepsilon^{-pt}\,dt$$

Let $y = t^n$. Then by the well-known integral, we have

$$\int_0^\infty t^n\varepsilon^{-pt}\,dt = \frac{1 \cdot 2 \cdot 3 \,.\,.\,.\, n}{p^{n+1}}$$

So $\dfrac{1}{pH(p)} = \dfrac{n!}{p^{n+1}}$ or $H(p) \cdot y = p^n \dfrac{t^n}{n!} = 1;\quad \therefore \dfrac{1}{p^n}(1) = \dfrac{t^n}{n!}$

and $$\frac{1}{p^{n-1}} = \frac{t^{n-1}}{(n-1)!} \text{ etc.}$$

As a further example, we have

$$\int_0^\infty \varepsilon^{-pt}\sin\lambda t\,dt = \frac{\lambda}{p^2+\lambda^2}$$

* *loc. cit.*

so that $1/H(p) = \lambda p/(p^2 + \lambda)$, which is the equivalent of $\sin \lambda t$. In Appendix I we have collected some of the more important results for ready reference.

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CHAPTER XI

CIRCUITS WITH VARIABLE PARAMETERS

General. The assumption of constant parameters does not in general agree with conditions obtaining in practice. For example, it is known that resistance varies with temperature and that the inductance of an iron-cored coil varies with the current. There are numerous other such effects. In subsequent discussion we segregate parameters as follows—

1. Those varying with respect to x, the independent variable, where integration is with respect to x. Such equations lead to solutions in a power series, of which the best known is the Bessel. The differential equation is of ordinary linear type with polynomial coefficients.

2. Those depending on the unknown y, the dependent variable. Such equations lead to a functional integral equation. As the solutions of these are given in the form of successive approximations, we will discuss them and pass to graphical methods.

3. Those depending on the unknown y, but the connecting function is complicated. Solutions can only be obtained graphically or by use of a machine integraph (see Bibliography, Chapter VI, for reference to the latter). In general it will be found that there is no great difficulty in setting up the equation, but the numerical solution of it is quite another matter. So, for engineering purposes at least, resort is had to graphical methods. These lead to no general result, as each problem must be treated anew.

1. **Power Series, Bessel Functions.** To introduce the first group we consider a line whose characteristic constants are a function of the line length, e.g. let $Rx = R_0/x$ and $Cx = C_0x$. With $G = L = 0$, we have

$$-\frac{dV}{dx} = RI = \frac{R_0 I}{x} \quad . \quad . \quad . \quad (452)$$

$$-\frac{dI}{dx} = CpV = C_0xpV \quad . \quad . \quad (453)$$

$$\therefore \quad \frac{d}{dx}\left[\frac{1}{R}\cdot\frac{dV}{dx}\right] = CpV$$

or
$$\frac{d^2V}{dx^2} + \frac{1}{x}\frac{dV}{dx} - k^2V = 0 \quad . \quad . \quad . \quad . \quad (454)$$
$$\therefore \quad V = AI_0(kx) + BK_0(kx)$$
where $k^2 = \frac{R_0}{x} C_0xp = R_0C_0p$, A and B are integration constants. The solution of this equation is given by a Bessel series.

2. **A Summary of Mathematical Results.** For an equation such as
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(k^2 - \frac{m^2}{x^2}\right) \quad . \quad . \quad . \quad (455)$$
where m is integral or nonintegral, it is shown in mathematical texts that the solution is given by
$$y = AJ_m(kx) + BY_m(xk) \quad (456)$$
where A and B are constants determinate from boundary conditions. Here $J_m(kx)$ denotes the series of the Bessel function of the *first kind* and mth order, and $Y_m(kx)$ that of the *second kind* and mth order. For the zeroth order we have the following expressions—
$$J_0\,(kx) = 1 - \frac{(kx)^2}{2^2} + \frac{(kx)^4}{2^24^2} - \frac{(kx)^6}{2^24^26^2} \quad . \quad . \quad (457)$$
when the argument is small. When it is large, then
$$J_0\,(kx) \cong \sqrt{\left(\frac{2}{\pi kx}\right)} \cos\,(kx - \pi/4) \quad . \quad . \quad (458)$$
For the second kind we get the following expressions—
$$Y_0\,(kx) = \frac{2}{\pi}\,(\alpha - \text{lgn}_e\, kx)\, J_0(kx) + \frac{(kx)^2}{2^2} - (1 + \tfrac{1}{2})\,\frac{(kx)^4}{2^24^2} \quad (459)$$
where $\alpha = \text{lgn}_e\, 2 - \gamma = 0{\cdot}11593$ for a small argument. When it is large, we have
$$Y_0\,(kx) \cong \sqrt{\left(\frac{2}{\pi kx}\right)} \sin\,(kx - \pi/4) \quad . \quad . \quad (460)$$

Generating Function. The form of the expression for the various orders may be developed from an expression
$$\varepsilon^{\frac{1}{2}x(t-1/t)} = \varepsilon^{\frac{1}{2}xt}\,.\,\varepsilon^{-x/2t} \quad . \quad . \quad . \quad (461)$$
On expansion we have
$$\left[1 + \frac{xt}{2} + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \quad . \; . \; .\right] . \left[1 - \frac{x}{2t} + \frac{1}{2!}\left(\frac{x}{(2t)}\right)^2 . \; . \; .\right]$$

Multiplying out and putting $t = 0$ gives the expansion for $J_0(x)$ as

$$1-\left(\frac{x}{2}\right)^2+\frac{x^4}{2^2 4^2}\ \cdot\ \cdot\ \cdot$$

Collecting the terms in t, we get the expansion

$$J_1(x)=\frac{x}{2}-\frac{x^3}{2^2 4}+\ \cdot\ \cdot\ \cdot$$

Or we have that

$$\varepsilon^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + t^2J_2(x) + tJ_{-1}(x)\ \ .\ \ .\ \ . \quad (462)$$

On substitution of $\varepsilon^{j\theta} = t$, the generating function becomes

$$\varepsilon^{jx\sin\theta} = J_0(x) + 2(J_2\cos 2\theta + J_4\cos 4\theta\ \ .\ \ .\ \ .) + 2j(J_1\sin\theta\ \ .\ \ .\ \ .)$$

Expanding and equating reals and imaginaries, we have

$$\cos(x\sin\theta) = J_0(x) + 2(J_2\cos 2\theta + J_4\cos 4\theta\ \ .\ \ .\ \ .)\ . \quad (463)$$

$$\sin(x\sin\theta) = 2\,(J_1\sin\theta + J_3\sin 3\theta\ \ .\ \ .\ \ .)\quad .\quad .\quad (464)$$

And it may be shown

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi}\varepsilon^{jx\sin\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\varepsilon^{jx\cos\theta}\,d\theta\quad .\quad (465)$$

or generally

$$J_m(x)=\frac{j^{-m}}{2\pi}\int_0^{\pi}\varepsilon^{jx\cos\theta}\cos m\theta\,d\theta\quad .\quad .\quad .\quad (466)$$

Differentiation. Differentiating $J_0(kx)$ with respect to (kx) gives

$$-\frac{kx}{2}+\frac{(kx)^3}{2^2 4}\ \ .\ \ .\ \ .=-J_1\,(kx)$$

If the differentiation is carried out with respect to x, we get

$$-\frac{k^2x}{2}+\frac{k^4x^3}{2^2 4}\ \ .\ \ .\ \ .=-kJ_1\,(kx)$$

So we get the *Recurrence* equations

$$\begin{aligned} xJ'_m(x) &= mJ_m(x) - xJ_{m+1}(x) \\ &= -\,mJ_m(x) + xJ_{m-1}(x) \end{aligned}\quad .\quad (467)$$

Modified Forms. For an equation in the form

$$\frac{d^2y}{dx}+\frac{1}{x}\frac{dy}{dx}-k^2y=0\ .\quad .\quad .\quad (468)$$

with $k = 1$, the solution is

$$y = AJ_0\,(jx) + BY_0\,(jx)$$

or, as usually written—

$$y = AI_0\,(x) + BK_0\,(x) \qquad . \qquad . \quad (469)$$

where I_0 is the modified form of J_0 and K_0 of Y_0.

For small arguments

$$I_0\,(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} \cdot \cdot \cdot = \frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-x\cos\theta}\,d\theta \quad (470)$$

and for large arguments

$$I_0\,(x) = \frac{\varepsilon^x}{\sqrt{(2\pi x)}}\left[1 + \frac{1}{8x} + \frac{1^2 3^2}{2\,!}\cdot\frac{1}{(8x)^2} + \cdot \cdot \cdot\right] \quad (471)$$

For series of the second kind, with small argument we get

$$K_0\,(x) = -\left(\gamma + \operatorname{lgn}\frac{x}{2}\right)I_0\,(x)$$
$$+ \frac{x^2}{2^2} + (1 + \tfrac{1}{2})\frac{x^4}{2^2 4^2} + \cdot \cdot \cdot \qquad . \quad (472)$$

and for large arguments we have

$$K_0\,(x) = \varepsilon^{-x}\sqrt{\left(\frac{\pi}{2x}\right)}\left[1 - \frac{1}{8x} + \frac{1^2 3^2}{2\,!}\cdot\frac{1}{(8x)^2} + \cdot \cdot \cdot\right] \quad (473)$$

The recurrence formula gives

$$I_{-m}(x) = I_m(x) : I_0'(x) = I_1(x)$$
$$K_{-m}(x) = K_m(x) : K_0'(x) = -K_1(x) \quad . \qquad . \quad (474)$$

OSCILLATING FUNCTIONS. For an equation such as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - jk^2y = 0 \qquad . \qquad . \quad (475)$$

the solution is given by

$$y = AJ_0\,(j^{\frac{3}{2}}kx) + BY_0\,(j^{\frac{3}{2}}kx) \quad . \qquad . \quad (476)$$

The generating function is now given by

$$\varepsilon^{j^{\frac{3}{2}}x(t-1/t)} = [1 + \tfrac{1}{2}j^{\frac{3}{2}}xt + \; . \; . \; .]\,[1 - \tfrac{1}{2}j^{\frac{3}{2}}x/t \; . \; . \; .]$$

So

$$J_0(j^{\frac{3}{2}}x) = \left[1 - \frac{(\frac{1}{2}x)^4}{2\,!} + \frac{(\frac{1}{2}x)^8}{4\,!} \; . \; . \; .\right]$$
$$+ j\left[(\tfrac{1}{2}x)^2 - \frac{(\frac{1}{2}x)^6}{3\,!} \; . \; . \; .\right] . \qquad . \quad (477)$$

or, as usually written, ber $x + j$ bei x . $\quad$. $\quad$. $\quad$. (478)

For functions of the second kind, we have

$$K_0\,(j^{\frac{3}{2}}x) = \operatorname{ker} x + j \operatorname{kei} x \quad . \quad . \quad (479)$$

where

$$\operatorname{ker} x = [\alpha - \operatorname{lgn} x] \operatorname{ber} x + (\pi/4) \operatorname{bei} x - (1+\tfrac{1}{2}) \frac{x^4}{2^2 4^2} \cdots \quad (480)$$

$$\operatorname{kei} x = (\alpha - \operatorname{lgn} x) \operatorname{bei} x - (\pi/4)$$

$$\operatorname{ber} x + \frac{x^2}{2^2} - (1 + \tfrac{1}{2} + \tfrac{1}{3}) \frac{x^6}{2^2 4^2 6^2} \cdots \quad (481)$$

Differentiating, we get

$$\frac{d}{d(kx)} J_0 = \operatorname{ber}'(kx) + j \operatorname{bei}'(kx)$$

also

$$\frac{dJ_0}{dx} = k\,(\operatorname{ber}' kx + j \operatorname{bei}' kx) \quad . \quad . \quad (482)$$

Approximations. When $x > 10$, we may use the following—

$$\left.\begin{aligned}
\operatorname{ber} x &= \left[\cos(x/\sqrt{2} - \pi/8) + \frac{1}{8x}\sin(x/\sqrt{2} + \pi/8)\right] \\
\operatorname{bei} x &= \left[\sin(x/\sqrt{2} - \pi/8) - \frac{1}{8x}\cos(x/\sqrt{2} + \pi/8)\right] \\
\operatorname{ber}' x &= \left[\cos(x/\sqrt{2} + \pi/8) - \frac{3}{8x}\cos(x/\sqrt{2} - \pi/8)\right] \\
\operatorname{bei}' x &= \left[\sin(x/\sqrt{2} + \pi/8) - \frac{3}{8x}\sin(x/\sqrt{2} - \pi/8)\right]
\end{aligned}\right\} \times \frac{1}{\sqrt{(2\pi x)}}\,\varepsilon^{x/\sqrt{2}} \quad . \quad . \quad (483)$$

$$\left.\begin{aligned}
\operatorname{ker} x &= \left[\cos(x/\sqrt{2} + \pi/8) - \frac{1}{8x}\sin(x/\sqrt{2} + \pi/8)\right] \\
\operatorname{kei} x &= \left[-\sin(x/\sqrt{2} + \pi/8) - \frac{1}{8x}\cos(x/\sqrt{2} + \pi/8)\right] \\
\operatorname{ker}' x &= \left[-\cos(x/\sqrt{2} - \pi/8) + \frac{3}{8x}\sin(x/\sqrt{2} - \pi/8)\right] \\
\operatorname{kei}' x &= \left[\sin(x/\sqrt{2} - \pi/8) + \frac{3}{8x}\cos(x/\sqrt{2} - \pi/8)\right]
\end{aligned}\right\} \times \sqrt{(\pi/2x)}\,.\,\varepsilon^{-x/\sqrt{2}}$$

When the arguments are very large, then the first term is a sufficient approximation.

Vector Diagram. An oscillating function may be represented by a right-angled triangle with the hypotenuse or modulus

$$M_0 \,\underline{/\phi_0} = \sqrt{(\operatorname{ber}^2 x + \operatorname{bei}^2 x)}$$

at an angle ϕ_0 to the base line, such that

$$\tan \phi_0 = (\text{bei}\, x)/(\text{ber}\, x)$$

The second kind may be similarly represented

$$N_0 \underline{/\theta_0} = \sqrt{(\text{ker}^2 x + \text{kei}^2 x)} \tan \theta = (\text{kei}\, x)/(\text{ker}\, x)$$

The following facts are useful in respect to differentiation—

$$\frac{d}{dx} \cdot J_0 (j^{\frac{3}{2}} x) = \text{ber}' x + j\, \text{bei}' x \quad . \quad . \quad (484)$$

$$\frac{1}{j^{\frac{3}{2}}} \frac{d}{dx} \cdot J_0 (j^{\frac{3}{2}} x) = j^{-\frac{3}{2}} [\text{ber}' x + j\, \text{bei}' x] = - J_1 (j^{\frac{3}{2}} x) \quad . \quad (485)$$

So we may write

$$\text{ber}' x + j\, \text{bei}' x = -j^{\frac{3}{2}} (M_1 \varepsilon^{j\phi_1}) \quad . \quad . \quad (486)$$

As $-\varepsilon^{-j\pi/4} = j^{\frac{3}{2}}$, we have that

$$\sqrt{(\text{ber}'^2 x + \text{bei}'^2 x)} = M_1 \underline{/(\phi_1 - \pi/4)} \quad . \quad (487)$$

Similarly we get that

$$\sqrt{(\text{ker}'^2 x + \text{kei}'^2 x)} = N_1 \underline{/(\theta_1 - \pi/4)} \quad . \quad (488)$$

Thus we get readily that

$$\text{ber}' x = M_1 \cos (\phi_1 - \pi/4), \; \text{bei}' x = M_1 \sin (\phi_1 - \pi/4)$$

and the following combinations may be developed—

$$\begin{aligned} &\text{ber}\, x \,.\, \text{bei}' x - \text{bei}\, x\, \text{ber}' x = M_0 M_1 \sin (\phi_1 - \phi_0 - \pi/4) \\ &\text{ber}\, x\, \text{ber}' x + \text{bei}\, x\, \text{bei}' x = M_0 M_1 \cos (\phi_1 - \phi_0 - \pi/4) \\ &\text{bei}' x\, \text{ker}' x - \text{ber}' x\, \text{kei}' x = M_1 N_1 \sin (\phi_1 - \theta_1) \qquad (489) \\ &\text{ber}' x\, \text{ker}' x + \text{bei}' x\, \text{kei}' x = M_1 N_1 \cos (\phi_1 - \theta_1) \end{aligned}$$

TABLES. Tables of Bessel functions will be found in several of the references to this chapter. We give below the roots of $J_m(x)$, where m is the order of the function and s that of the root.

s	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
1	2·404	3·832	5·135	6·379	7·586	8·780
2	5·520	7·016	8·417	9·760	11·064	12·339
3	8·654	10·173	11·620	13·017	14·373	15·700
4	11·792	13·323	14·796	16·224	17·616	18·982
5	14·931	16·470	17·960	19·410	20·827	22·220
6	18·072	19·616	21·117	22·583	24·018	25·431
7	21·212	22·760	24·270	25·749	27·200	28·628
8	24·353	25·903	27·421	28·909	30·371	31·813
9	27·494	29·047	30·570	32·050	33·512	34·983

Mental picture. A general idea of the form of these functions is as follows—

(i) $I_0(x)$ with real argument is of the same shape as ε^x, with the exception that at $x = 0$, $I_0(x) = 0$; and at $x = \infty$, $I_0(x) = \infty$.

(ii) $K_0(x)$ behaves as ε^{-x}, with the restriction that at $x = 0$ it is infinite, falling to zero at $x = \infty$. Functions of this type are frequently eliminated by boundary conditions. Recalling the definitions of the incident and reflected waves, we see that $I_0(x)$ is the reflected and $K_0(x)$ the incident wave.

(iii) $J_0(x)$ approximates to a cosine wave.

(iv) $Y_0(x)$ approximates to a sine wave.

From the table of roots it is seen that the periods of these two functions are both 2π when the value of x is above 10.

Operational Expressions. As we treat the problems of this chapter by formal mathematical methods, we will now show some operational expressions which lead to the Bessel series. For example, we have on expansion as a convergent series the following expression—

$$\frac{\Delta}{\sqrt{(\Delta^2 - a^2)}} = \frac{1}{\sqrt{(1 - a^2/\Delta^2)}} = 1 + \frac{1}{2}\frac{a^2}{\Delta^2} + \frac{1\cdot 3}{2^2}\cdot\frac{a^4}{\Delta^4} + \dots$$

$$= 1 + \frac{1}{2^2}a^2x^2 + \frac{1\cdot 3}{2^2.4^2}a^4x^4 \dots$$

$$= I_0\,(ax)^*$$

On expanding the expression—

$$\left(\frac{\Delta}{\Delta + 2\alpha}\right)^{\frac{1}{2}} = \sqrt{\frac{\Delta}{2a}}\left(1 + \frac{\Delta}{2a}\right)^{-\frac{1}{2}} = \left[1 - \frac{\Delta}{2a} + \frac{1\cdot 3}{2!}\cdot\left(\frac{\Delta}{2a}\right)^2 \dots\right]\sqrt{\frac{\Delta}{2a}}$$

Now,
$$\sqrt{\frac{\Delta}{2\alpha}} \doteq \frac{1}{\sqrt{(2a)}}\cdot\frac{1}{\sqrt{(\pi x)}}.$$

$$\therefore \left(\frac{\Delta}{\Delta + 2a}\right)^{\frac{1}{2}} \doteq \frac{1}{\sqrt{(2\pi a x)}}\left[1 + \frac{1^2}{8ax} + \frac{1^2.3^2}{2!(8ax)^2} + \dots\right]$$

$$\doteq \varepsilon^{-ax} I_0(ax) \dots \dagger$$

Now, $\varepsilon^{ax}\left(\dfrac{\Delta}{\Delta + 2a}\right)^{\frac{1}{2}} = \left(\dfrac{\Delta - \alpha}{\Delta + \alpha}\right)^{\frac{1}{2}}\varepsilon^{ax} = \dfrac{\Delta}{\sqrt{\Delta^2 - \alpha^2}}$, which is the expression at the start.

* See equation (470). † See equation (471).

If now we substitute $a^2\left(1 - \frac{x^2}{v^2t^2}\right)$ for a^2, then

$$\frac{p}{\sqrt{[p^2 - a^2 (1 - x^2/v^2t^2)]}} = I_0\left(a\left[t^2 - \frac{x^2}{v^2}\right]^{\frac{1}{2}}\right)$$

An expression such as

$$\frac{2\Delta}{\sqrt{(\alpha^2 - \Delta^2)}} = \varepsilon^{-\alpha x} . \varepsilon^{\alpha x} . \frac{2\Delta}{\sqrt{(\alpha^2 - \Delta^2)}}$$

$$= 2\varepsilon^{-\alpha x} . \frac{\Delta - \alpha}{\sqrt{(2\alpha\Delta - \Delta^2)}} . \frac{\Delta}{\Delta - \alpha}$$

By expansion and integration, this becomes

$$\varepsilon^{-\alpha x}\left[\frac{2}{\pi a x}\right]^{\frac{1}{2}} . \left[1 - \frac{1}{8ax} + \frac{1^2 . 3^2}{2\,!(8ax)^2} \ . \ . \ . \ \right] = \frac{\pi}{2} K_0\,(ax).$$

Further expressions are given in Appendix I.

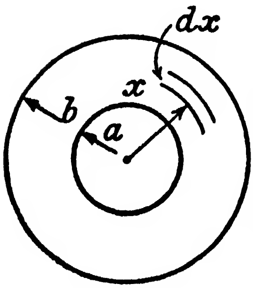

FIG. 47

3. **Skin Effect in Isolated Tube.** Reverting to the problem discussed in Chapter IX, we will here take into account the field set up by eddy currents. For a tubular conductor as shown (Fig. 47), the voltage across the ends of an elementary tube of radius x and thickness dx is

$$e = Ri + \frac{d\phi}{dt}$$

But $Ri = \frac{\rho l}{2\pi x\, dx}\, \sigma\, (2\pi x\, dx) = \rho\sigma$ per cm. length.

So
$$e = \rho\sigma + \frac{d\phi}{dt}$$

where ρ is the resistivity and σ the current density. The disturbing flux is $\phi = \int_x^b \frac{2I_x}{x}\, dx$

where I_x is the current enclosed between radii a and x. So

$$\frac{d\phi}{dx} = -\frac{2I_x}{x} = -\frac{2}{x}\int_a^x 2\pi\sigma x\, dx$$

On differentiating with respect to x, since the p.d.s across all tubes must be the same, we get

$$\rho\frac{d\sigma}{dx} - \frac{d}{dt} . \frac{4\pi}{x} . \int_a^x \sigma x\, dx = 0$$

Removing the integral sign by a second differentiation, we have

$$\frac{d^2\sigma}{dx^2} + \frac{1}{x}\frac{d\sigma}{dx} = \frac{4\pi}{\rho}\frac{d\sigma}{dt} \quad . \quad . \quad . \quad (490)$$

When σ varies harmonically with time, we have

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\sigma}{dx}\right) = \frac{4\pi}{\rho}j\omega\sigma = jm^2\sigma$$

where $m^2 = 4\pi\omega/\rho$.

The solution of this equation is

$$\sigma = AJ_0\,(j^{\frac{3}{2}}mx) + BK_0\,(j^{\frac{1}{2}}mx) \quad . \quad . \quad . \quad (491)$$

Usually, this equation is written in the form of oscillating functions, thus—

$$\sigma = A[\text{ber } mx + j \text{ bei } mx] + B\,[\text{ker } mx + j \text{ kei } mx] \quad . \quad (492)$$

A and B are constants determinate from boundary conditions.

Determination of A and B. For an isolated tube, at $x = a$, then $I_x = 0$ and $d\sigma/dx = 0$. This condition gives

$$A[\text{ber}'\,ma + j\text{ bei}'\,ma] + B\,[\text{ker}'\,ma + j\text{ kei}'\,ma] = 0$$

or

$$\frac{B}{A} = -\frac{\text{ber}'\,ma + j\text{ bei}'\,ma}{\text{ker}'\,ma + j\text{ kei}'\,ma} \quad . \quad . \quad (493)$$

Also the total current in the tube is $I = 2\pi\int_a^b x\sigma\,dx$.

$$\therefore \quad I = 2\pi\int_a^b x[AJ_0 + BK_0]dx$$

Now

$$\int x\text{ ber } mx\,.\,dx = (x/m)\text{ bei}'\,mx$$

$$\int x\text{ bei } mx\,.\,dx = -\,(x/m)\text{ ber}'\,mx$$

So

$$I = 2\pi\left[\frac{x}{m}A\,(\text{bei}'\,mx - j\text{ ber}'\,mx) + \frac{x}{m}B\,(\text{kei}'\,mx - j\text{ ker}'\,mx)\right]_a^b \quad . \quad (494)$$

From equations (493) and (494) we get

$$\left.\begin{aligned} A &= \frac{mI}{2\pi b}\,(\text{ker}'\,ma + j\text{ kei}'\,ma)/\Delta\,(mb) \\ -B &= \frac{mI}{2\pi b}\,(\text{ber}'\,ma + j\text{ bei}'\,ma)/\Delta\,(mb) \end{aligned}\right\} \quad . \quad (495)$$

where

$$\Delta\,(mb) = (\text{bei}'\,mb - j\,\text{ber}'\,mb)\,(\text{ker}'\,ma + j\,\text{kei}'\,ma) - (\text{kei}'\,mb - j\,\text{ker}'\,mb)\,(\text{ber}'\,ma + j\,\text{bei}'\,ma).$$

For any numerical problem the values of A and B may be evaluated with the help of tables.

RESISTANCE AND INDUCTANCE. When the current is uniformly distributed, the resistance per centimetre length will be $\rho/\pi(b^2 - a^2)$. At $x = b$ the disturbing flux is zero, and the voltage drop is given by $Z_0\sigma_0$. Due to the voltage drops being equal, we have

$$Z_0\sigma_0 = Z_x\sigma_x = IR_{DC} = IZ' \text{ (say)}$$

So $$Z' = \frac{Z_0\sigma_0}{I} = \frac{\rho\sigma_0}{I} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (496)$$

Substituting, we get

$$Z' = \frac{\rho m}{2\pi}\left\{\frac{A\,(\text{ber}\,mb + j\,\text{bei}\,mb) + B\,(\text{ker}\,mb + j\,\text{kei}\,mb)}{[Ax\,(\text{bei}'\,mx - j\,\text{ber}'\,mx) + Bx\,(\text{kei}'\,mx - j\,\text{ker}'\,mx)]_a^b}\right\}$$

Let $\chi = B/A$.

Then we have

$$Z' = j\frac{\rho m}{2\pi b}\left[\frac{(\text{ber}\,mb + j\,\text{bei}\,mb) + \chi\,(\text{ker}\,mb + j\,\text{kei}\,mb)}{(\text{ber}'\,mb + j\,\text{bei}'\,mb) + \chi\,(\text{ker}'\,mb + j\,\text{kei}'\,mb)}\right]$$

or

$$\frac{Z'}{R_{DC}} = \frac{jmb}{2}\,\frac{b^2 - a^2}{b^2}$$

$$\left[\frac{(\text{ber}\,mb + j\,\text{bei}\,mb) + \chi\,(\text{ker}\,mb + j\,\text{kei}\,mb)}{(\text{ber}'\,mb + j\,\text{bei}'\,mb) + \chi\,(\text{ker}'\,mb + j\,\text{kei}'\,mb)}\right] \quad . \quad (497)$$

The real part of this expression gives the resistance, and the imaginary part the inductance. For further development we note from tables that, when the argument is in the neighbourhood of 8, the "ker" and "kei" terms are small and may be neglected. So the expression for the impedance becomes

$$\frac{Z'}{R_{DC}} = j\frac{mb}{2}\left[1 - \frac{a^2}{b^2}\right]\frac{J_0}{J_0'} = j\frac{mb}{2}\left[1 - \frac{a^2}{b^2}\right]\frac{M_0}{M_1}\underline{|\phi_0 - \phi_1 + \pi/4}$$

$$\therefore \frac{R}{R_{DC}} = \frac{mb}{2}\left(1 - \frac{a^2}{b^2}\right)\frac{M_0}{M_1}\cos\,(\phi_0 - \phi_1 + 3\pi/4) \quad . \quad . \quad (498)$$

and

$$\frac{X}{R_{DC}} = \frac{mb}{2}\left(1 - \frac{a^2}{b^2}\right)\frac{M_0}{M_1}\sin\,(\phi_0 - \phi_1 + 3\pi/4) \quad . \quad . \quad (499)$$

And for very large values of mb we get,

$$\frac{R}{R_{DC}} = \frac{mb}{2\sqrt{2}}\left(1 - \frac{a^2}{b^2}\right)$$

$$\frac{X}{R_{DC}} = \frac{mb}{2\sqrt{2}}\left(1 - \frac{a^2}{b^2}\right)$$

4. **Solid Conductor.** Inserting $a = 0$, we see that B must be zero, and the appropriate equation is

$$\sigma_x = AJ_0\,(mxj^{\frac{3}{2}})$$

If σ_0 is the density at $x = b$, we have

$$\sigma_0 = AJ_0\,(mbj^{\frac{3}{2}})$$

$$\therefore \quad \sigma_x = \sigma_0 \frac{J_0\,(mx)}{J_0\,(mb)}$$

or in terms of the moduli, we get

$$\sigma_x = \sigma_0 \frac{M_0\,(mx)}{M_0\,(mb)} \underline{|\,(\phi_0\,(mx) - \phi_1\,(mb)} \qquad (500)$$

The current density is $\dfrac{M_0\,(mx)}{M_0\,(mb)}$ times the density at the surface, and the angle of lag is $\theta_0 - \theta_1$. The ratio $\dfrac{M_0\,(mx)}{M_0\,(mb)}$ is known as the penetration coefficient.

The value of the current flowing in the wire is $2\pi\int_0^b \sigma x\,dx$.

$$\therefore \quad I = \frac{2\pi\sigma_0}{J_0\,(mb)}\int_0^b J_0\,(mx)\,xdx = \frac{2\pi\sigma_0 b}{mj^{\frac{3}{2}}}\,\frac{J_1(mb)}{J_0(mb)} \quad . \quad . \quad (501)$$

The average density is

$$\sigma_{av} = \frac{I}{\pi b^2}$$

$$= \frac{2\sigma_0}{mb}\left(\frac{\text{bei}'\,mb - j\,\text{ber}'\,mb}{\text{ber}\,mb + j\,\text{bei}\,mb}\right)$$

So $\quad \sigma_x = \sigma_{av}\dfrac{mb}{2}\cdot\dfrac{M_0\,(mx)}{M_1\,(mb)}\,\underline{|\phi_0\,(mx) - \phi_1\,(mb) + 3\pi/4}$

or $\quad \sigma_x = \sigma_{av}\dfrac{mb}{2}\left(\dfrac{\text{ber}\,mx + j\,\text{bei}\,mx}{\text{bei}'\,mb - j\,\text{ber}'\,mb}\right) \quad . \quad . \quad . \quad (502)$

Resistance and Inductance. As for the previous example, the resistance and inductance may be estimated by equating

the real and imaginary parts af the voltage drop. Thus in any elementary tube the voltage drop is $e_x = \rho\sigma_x + \frac{d\phi_x}{dt}$, and for harmonically varying currents we get $e = \rho\sigma_x + j\omega\phi_x$. At the outside layer the disturbing flux is zero, so

$$e = \rho\sigma_0 = I\,[R_e + jX_e)$$

where R_e and X_e are the effective resistance and inductance. As I is equal to the average density times the area, we get

$$R_e + jX_e = \frac{\rho\sigma_0}{\pi b^2 \sigma_{av}} = \frac{\rho}{\pi b^2}\cdot\frac{mb}{2}\left(\frac{\text{ber } mb + j \text{ bei } mb}{\text{bei}' mb - j \text{ ber}' mb}\right) \quad . \quad (503)$$

On equating the reals and imaginaries in this expression, we get

$$R_e = \frac{\rho}{\pi b^2}\cdot\frac{mb}{2}\left(\frac{\text{ber } mb \text{ bei}' mb - \text{bei } mb \text{ ber}' mb}{\text{ber}'^2 mb + \text{bei}'^2 mb}\right) \quad . \quad (504)$$

$$L_e = \frac{\rho}{\pi b^2}\cdot\frac{mb}{2}\left(\frac{\text{ber } mb \text{ ber}' mb + \text{bei } mb \text{ bei}' mb}{\text{ber}'^2 mb + \text{bei}'^2 mb}\right) \quad . \quad (505)$$

The latter expression for L_e is due to internal linkages only. These expressions may be further simplified.* It can be shown that for small values of the argument

$$R_e/R_{DC} = 1 + \frac{1}{12}\left(\frac{mb}{2}\right)^2 - \frac{1}{180}\left(\frac{mb}{2}\right)^8 \ldots \quad (506)$$

$$\frac{L_{AC} - 2\log_e (s/b)}{L_{DC} - 2\log (s/b)} = \frac{2}{mb}\left[\frac{1}{\sqrt{2}} - \frac{3}{8\sqrt{2}m^2b^2} - \frac{3}{8m^2b^2} \ldots\right] \quad (507)$$

Losses in Wire. The eddy current losses in a wire are given by I^2R_e, and on substituting we get

$$W = \frac{\pi\sigma_0{}^2\rho b}{m}\cdot\frac{M_1\,(mb)}{M_0\,(mb)}\cos\,(\phi_0 - \phi_1 + 3\pi/4) \quad . \quad (508)$$

5. **Bessel Cable.** Heaviside† considers a cable whose constants are a function of the length. Thus $1/R_x = x/R_0$ and $C_x = C_0 x$. The equation is

$$\frac{1}{x}\cdot\frac{d}{dx}\left(\frac{1}{R}\,\frac{dV}{dx}\right) = m^2 V$$

or

$$V = AI_0\,(mx) + BK_0\,(mx) \quad . \quad . \quad (509)$$

* Russell: *Phil. Mag.* (1909), Vol. 17, p. 524.

† O. Heaviside: *Electro-magnetic Theory*, Vol. II, p. 212.

On charging an empty cable we get

$$V = AI_0\,(mx) \quad . \quad . \quad . \quad . \quad . \quad (510)$$

$$I = -\frac{1}{R}\frac{dV}{dx} = -A\frac{m}{R}I_0{}'\,(mx) \quad . \quad . \quad (511)$$

To determine the integration constant, let the voltage at $x = l$ be E_0, then

$$V = E_0\frac{I_0\,(mx)}{I_0\,\,(ml)} \quad . \quad . \quad . \quad . \quad (512)$$

and

$$I = -\frac{mx}{R_0}\,[I_0{}'\,(mx)]\,A \quad . \quad . \quad . \quad (513)$$

By the recurrence formula we get

$$I_x = -\frac{mx}{R_0}\frac{I_1\,(mx)}{I_0\,\,(ml)}E_0$$

Denoting $mxj^{\frac{3}{2}}$ by u for harmonic quantities, we have $p = j\omega$. With large arguments we have

$$J_n\,(u) = \sqrt{\left(\frac{2}{\pi u}\right)}\cos\left(u - \frac{2n+1}{4}\pi\right)$$

So

$$I_x = -\frac{mx}{R_0}\sqrt{\left(\frac{l}{x}\right)}\cdot\frac{\cos\,(u_x - 3\pi/4)}{\cos\,(u_l - \pi/4\,)}E_0 \quad . \quad (514)$$

$$= \frac{mx}{R_0}\sqrt{\left(\frac{l}{x}\right)}\cdot\frac{\cos u_x - \sin u_x}{\cos u_l + \sin u_l}\cdot E_0$$

At $x = l$, $I_l = \dfrac{ml}{R_0}\left(\dfrac{1 - \sin 2u_l}{\cos 2u_l}\right)E_0$. . . . . (515)

Let $\alpha_0 = \sqrt{(R_0C_0)}$. Then $m = j\alpha_0 p^{\frac{1}{2}}$. On substituting, we get

$$I_l = j\frac{\alpha_0 lp^{\frac{1}{2}}}{R_0}\left(\frac{1 - j\sinh 2j\alpha_0 lp^{\frac{1}{2}}}{\cosh 2j\alpha_0 lp^{\frac{1}{2}}}\right)E_0 \quad . \quad . \quad (516)$$

Dealing with real terms only, and noting that tanh approaches unity for large angles, we have

$$I_l = \frac{\alpha_0 lp^{\frac{1}{2}}}{R_0}E_0 = \frac{E_0\alpha_0 l}{R_0}\cdot\frac{1}{\sqrt{(\pi t)}} \quad . \quad . \quad . \quad (517)$$

Expansion by Operational Processes. With a voltage equation we have that

$$V = \frac{I_0\,(mx)}{I_0\,(ml)}E_0$$

The determinantal equation is $I_0(ml) = 0$. Applying the expansion theorem, we get

$$V = E_0 \left[1 + \Sigma \frac{I_0(mx)\, \varepsilon^{pt}}{p\frac{d}{dp} I_0(ml)} \right] \quad . \quad . \quad (518)$$

Let $m^2 = - s^2 = R_0C_0p$.

Then $$-\frac{ds}{dp} = R_0C_0/(2s)$$

So that $$p\frac{ds}{dp} = \frac{s}{2}$$

Again $$\frac{d}{dp} \cdot J_0(slj^{\frac{3}{2}}) = \frac{ds}{dp} \times lj^{\frac{3}{2}} \cdot J_0'(slj^{\frac{3}{2}})$$

So $$V = E_0 \left[1 + \Sigma \frac{J_0(sx)\, \varepsilon^{pt}\, \varepsilon^{-j3\pi/4}}{\frac{1}{2} sl\, J_0'(slj^{\frac{3}{2}})} \right] \quad . \quad . \quad (519)$$

The value of the roots of the equation $J_0(slj^{\frac{3}{2}}) = 0$ must now be obtained. Analytically the work is complicated, but for numerical work the roots are readily obtained from tables.

It is interesting to note that submarine cables are taper loaded. Thus the Newfoundland—Azores cable is loaded by mumetal wound directly on the conductors in a close continuous wrapping. An inductance of 0·056 to 0·230 henry per mile is obtained. It is claimed that tapering (i) reduces the reflection losses from the non-loaded part, and (ii) permits the use of lower-permeability materials for loading.

For further discussion reference should be made to Starr.*

6. **Filter Circuits.** This problem has been discussed in Chapter VII, but as Bessel functions give a more compact solution for calculations we will now develop the theory. For a steady voltage E, the response is given by

$$i_n = \frac{2E}{nL} \sum_{s=1}^{s=\infty} \frac{\sin \beta_s t}{\beta_s \cos s\pi}$$

when the effect of resistance is neglected.

* STARR: *Phil. Mag.* (1934), Vol. 17, p. 83.

We will now write it as

$$i_n = \frac{2E}{nL} \sum_{1}^{\infty} \int_0^t \frac{\cos \beta_s t}{\cos s\pi} dt \quad . \quad . \quad . \quad . \quad . \quad (520)$$

where $\beta_s = 2v \sin \frac{s\pi}{2n}$ and the step in the summation is $\pi/2n$.

Let $\theta = \frac{s\pi}{2n}$, $d\theta = \frac{\pi}{2n}$, and $2n\theta = s\pi$.

So with proper limits we get

$$i_n = \frac{4E}{\pi L} \int_0^t \int_0^{\pi/2} \frac{\cos (2vt \sin \theta) \cos 2n\theta}{\cos^2 s\pi} dt\, d\theta \quad . \quad . \quad (521)$$

Now, $\cos s\pi = (-1)^s$; then $\cos^2 s\pi = 1$.

The Bessel generating function shows that

$$\cos (2vt \sin \theta) = J_0 (2vt) + 2 [J_2 (2vt) \cos 2\theta + J_4 (2vt) \cos 4\theta \ldots] \quad (522)$$

All the terms vanish at the limits of integration with the exception of

$$\int_0^{\pi/2} 2J_{2n} (2vt) \cos^2 2n\theta d\theta = \frac{\pi}{2} . J_{2n} (2vt)$$

So that we get

$$i_n = \frac{2E}{L} \int_0^t J_{2n} (2vt)\, dt = \frac{2E}{L} \int_0^t \sum_{s=0}^{\infty} \frac{(-1)^s . (vt)^{2n+2s}}{s! \, \Gamma(2n + s + 1)} dt$$

$$= \frac{2E}{vL} \frac{2n + s + 1}{2n + 2s + 1} J_{2n+1} (2vt). \quad . \quad . \quad . \quad . \quad (523)$$

$$= \frac{2E}{vL} . C_s . J_{2n+1} (2vt). \text{ say.}$$

For the current in the rth mesh we get

$$i_r = \frac{2E}{nL} \sum \frac{\cosh (n - r)\alpha \sin \beta_s t}{\beta_s \cos s\pi} \quad . \quad . \quad . \quad . \quad (524)$$

$$= \frac{4E}{\pi L} \int_0^{\pi/2} \int_0^t \cos (2vt \sin \theta) \cos 2n\theta \cos (n - r)\, 2\theta\, d\theta\, dt$$

With the same discussion as above we note that

$$\cos 2n\theta . \cos (n - r) 2\theta = \cos 2n\theta (\cos 2n\theta \cos 2r\theta + \sin 2n\theta \sin 2r\theta)$$

But $\cos^2 2n\theta = \cos^2 s\pi = 1$, and $\sin 4n\theta = \sin 2s\pi = 0$. So we get the effective integral as

$$\int_0^{\pi/2} J_{2r}(2vt)\cos^2 2r\theta = \frac{\pi}{4} J_{2r}(2vt)$$

Hence

$$i_r = \frac{2E}{vL}\int_0^t J_{2r}(2vt)\,dt = \frac{2E}{vL} J_{2r+1}(vt)\,.\,C_r \qquad . \qquad . \quad (525)$$

So to find the current in any mesh all that is necessary is to alter the order of the Bessel function. Reference should be made to a paper by van der Pol,* wherein these results are obtained otherwise.

7. **Impedance of an Earth Return.** In order to calculate the short-circuit fault current of a line to earth, it is necessary to know the inductance of the earth return. This problem may be solved approximately by considering all the earth current to return within a circular trench, the radius of the trench to be equal to the height of the conductor above ground. We consider that the flow of the earth current is parallel everywhere to the axis of the trench and that the magnetic lines are centred on that axis.

If σ is the density, then $I \equiv \int_0^h \pi\sigma x\,dx$. Along the trench we have $\rho\sigma + \dfrac{d\phi}{dt} = e$. But e is independent of r, and so

$$\rho\frac{\partial\sigma}{\partial r} + \frac{\partial\phi}{\partial r\,\partial t} = 0$$

Moreover $\dfrac{\partial\phi}{\partial r} = -\dfrac{2}{r}\displaystyle\int_0^h \pi r\sigma\,dr$.

$$\therefore \qquad \rho\frac{\partial\sigma}{\partial r} = \frac{\partial}{\partial t}\cdot\frac{2\pi}{r}\int_0^h r\sigma\,dr$$

or

$$\rho r\frac{\partial^2\sigma}{\partial r^2} + \rho\frac{\partial\sigma}{\partial r} = \frac{\partial}{\partial t}(2\pi r\sigma)$$

$$\therefore \qquad \frac{\partial^2\sigma}{\partial r^2} + \frac{1}{r}\frac{\partial\sigma}{\partial r} = \frac{2\pi}{\rho}\,.\,j\omega\sigma$$

when harmonic quantities are considered.

* VAN DER POL: *Journ. I.E.E.* (1937), Vol. 81, p. 387.

Hence $$\sigma = A \,.\, J_0\,(mrj^{\frac{3}{2}}) + BK_0\,(mrj^{\frac{1}{2}})$$

At $r = \infty$, $\sigma = 0$. So $A = 0$.

This gives $$\sigma = BK_0\,(mrj^{\frac{1}{2}})$$

Again, the total earth current

$$I = \int_0^h \pi r\sigma\,dr$$

$$= \pi B \int_0^h r\,K_0\,(mrj^{\frac{1}{2}})\,dr$$

$$= \frac{\pi B}{mj^{\frac{1}{2}}}\left[h\,K_1\,(mhj^{\frac{1}{2}})\right]$$

So we have

$$\sigma = \frac{I\,mj^{\frac{1}{2}}}{\pi h}\left[\frac{K_0\ (mrj^{\frac{1}{2}})}{K_1\ (mhj^{\frac{1}{2}})}\right]$$

When we consider the voltage at $r = h$, then $E = \rho\sigma$.

So $$E = \frac{I\rho mj^{\frac{1}{2}}}{\pi h}\left[\frac{K_0}{K_1}\right](mhj^{\frac{1}{2}}).$$

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CHAPTER XII

CIRCUITS WITH VARIABLE PARAMETERS (*contd.*)

1. **Methods of Successive Approximations. General.** We have already seen that when the circuit parameters depend on the independent variable the solution may be obtained as a power series. We now investigate the matter further, for the case where the operational impedance can be considered as consisting of two components, one which is constant with respect to time, viz. $[Z(p)_0]$ (i.e. t is the independent variable), and one which varies with time $Z(p)_t$. Thus we have the equation

$$[Z(p)_0 + Z(p)_t]i = E$$

$$\therefore \quad i(t) = \frac{E}{Z(p)_0} - \frac{Z(p)_t \,.\, i_0(t)}{Z(p)_0} \qquad . \quad (526)$$

The first term of the right-hand side of equation (526) is the current due to an impedance which is invariable with respect to time, while the second term is due to the variable impedance. It is observed that the resultant current is less than the constant term. On writing

$$E/Z(p)_0 = i_0(t)$$

we may rewrite the equation as

$$i(t) = \frac{i_0(t)}{1 + \dfrac{Z(p)_t}{Z(p)_0}}$$

By expansion this gives

$$i(t) = (1 - a + a^2 \;.\;.\;.)i_0(t) \qquad . \qquad . \quad (527)$$

where $a = \dfrac{Z(p)_t}{Z(p)_0}$.

So we have

$$i(t) = i_0(t) - i_1(t) + i_2(t) \;.\;.\;. \qquad . \quad (528)$$

where

$$i_1(t) = \frac{Z(p)_t}{Z(p)_0} \,.\, i_0(t)$$

$$i_2(t) = \frac{Z(p)_t}{Z(p)_0} \,.\, i_1(t) \qquad . \qquad . \qquad . \quad (529)$$

$$\ldots \qquad \ldots \qquad \ldots$$

Now, $Z(p)_t \,.\, i_0(t)$ is the voltage across the variable component with a current $i_0(t)$ flowing. We will denote this by $v(t)$; $1/Z(p)_0 = A(t)$ is the admittance.

From the discussion on the Duhamel integral it is seen that $i_1(t)$ may be written as

$$i_1(t) = \frac{d}{dt}\int_0^t v(\lambda)\,.\,A(t-\lambda)d\lambda \quad . \quad . \quad . \quad (530)$$

If, for example, resistance be the variable component of $Z(p)_t$, then we have

$$i_1(t) = \frac{d}{dt}\cdot\int_0^t [r(\lambda)\,.\,I_0(\lambda)]\,.\,A(t-\lambda)\,.\,d\lambda$$

.. ..

.. ..

$$i_n(t) = \frac{d}{dt}\int_0^t [r(\lambda)\,.\,I_{n-1}(\lambda)]\,.\,A(t-\lambda)d\lambda \quad . \quad (531)$$

This method is not adapted to numerical work; but, in spite of the wearisome mathematics, it does give valuable information respecting the character of the oscillations.

2. A Variable-capacity Circuit. To demonstrate the above method we will consider an LC circuit wherein the capacity of the condenser is varied by means of a cam as shown in Fig. 48. Then the equation is

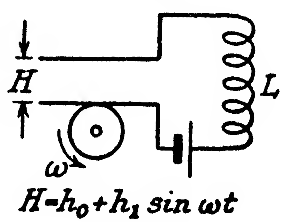

FIG. 48

$$Lpi + \frac{i}{pC} = E$$

Here $\dfrac{1}{C} = K[h_0 + h_1 \sin \omega t] = S_0 + S_1 \sin \omega t = S_0 + S_1(t)$.

So that $$E = Lpi + [S_0 + S_1(t)]\,.\,\frac{i}{p}$$

or $$i = \frac{E - S_1(t)\,.\,(i/p)}{Lp + (S_0/p)}$$

and $$i(t) = i_0(t) - \frac{S_1(t)\,.\,(i/p)}{Z(p)_0} \quad . \quad . \quad . \quad (532)$$

where $i_0(t)$ is the current flowing when the condenser plates are fixed at $h = h_0$. This equation may be written as

$$i(t) = \frac{i_0(t)}{1 + \frac{S_1(t)}{Z(p)} \cdot \frac{1}{p}} = \left\{1 - \frac{S_1(t)}{Z(p)} \cdot \frac{1}{p} + \left[\frac{S_1(t)}{Z(p)} \cdot \frac{1}{p}\right]^2 \cdot \cdot \cdot\right\} i_0(t)$$

i.e. $i(t) = i_0(t) - i_1(t) + i_2(t) \ldots$ . . . . (533)

where $i_1(t) = \frac{S_1(t)}{Z(p)} \cdot \frac{1}{p} \cdot i_0(t) = \frac{d}{dt}\int_0^t S_1(\lambda) \cdot \frac{i_0(\lambda)}{p} \cdot A(t-\lambda) \cdot d\lambda$

etc. . . . .

We will now develop a few terms of this approximation. For the first term we have

$$i_0(t) = \frac{Ep}{Lp^2 + S_0} \doteq \frac{E}{L\beta} \sin \beta t$$

where $\beta^2 = -\frac{S_0}{L}$.

We may consider $S_1(\lambda) \cdot \frac{i_0(\lambda)}{p}$ as a voltage $v(\lambda)$, in order to bring our expressions into line with the previous discussion of the Duhamel theorem. Here, then—

$$\begin{aligned} v_1(\lambda) &= S_1 \sin \omega\lambda \int_0^t i_0(\lambda) d\lambda \\ &= S_1 \sin \omega\lambda \cdot \frac{E}{L\beta^2} \cdot (1 - \cos \beta\lambda) \end{aligned}$$

Also $$A(t) = \frac{1}{L\beta} \cdot \sin \beta t$$

As $A(0) = 0$, we may write

$$\begin{aligned} i_1(t) &= \int_0^t v(\lambda) \cdot A'(t-\lambda) d\lambda \\ &= \frac{S_1 E}{L^2\beta^2} \cdot \int_0^t \sin \omega\lambda \cdot (1 - \cos \beta\lambda) \sin \beta(t-\lambda) \cdot d\lambda \end{aligned}$$

From which we get

$$\begin{aligned} i_1(t) = \frac{S_1 E}{2L^2\beta^2}\Bigg[&-\frac{\beta}{\omega^2 - \beta^2} \sin \omega t - \frac{2\beta^2(\omega^2 + 2\beta^2)}{\omega(\omega^2 - \beta^2)(\omega^2 - 4\beta^2)} \sin \beta t \\ &+ \frac{\beta}{\omega(\omega - 2\beta)} \sin (\omega - \beta)t + \frac{\beta}{\omega(\omega + 2\beta)} \sin (\omega + \beta)t\Bigg] \end{aligned}$$

Further development is cumbersome, but inspection of results reveals that $i(t)$ is multiperiodic.

3. **Helmholtz's Method and Taylor's Series.** We now give two methods which are used frequently to give solutions in the form of successive approximations. The first of these is the method of Helmholtz, and is best illustrated by an example, as follows. Let

$$p^2x + n^2x + hx^2 = a \cos \omega t \qquad . \quad . \quad (534)$$

with the boundary conditions $t = 0$, $x = c$, and $px = 0$.

Let us now seek a solution in the form

$$x = c\phi_1(t) + c^2\phi_2(t) \ldots \qquad . \quad . \quad (535)$$

Put $a = a_1c$.

On substituting in equation (534), and equating like powers of c in both equations, we have

$$\left.\begin{aligned} p^2\phi_1 + n^2\phi_1 \qquad &= a_1 \cos \omega t \\ p^2\phi_2 + n^2\phi_2 + h\phi_1{}^2 &= 0 \end{aligned}\right\} \quad . \quad . \quad (536)$$

etc. . . . .

By integration of these auxiliary equations, we get for the first that

$$\phi_1 = A \cos nt + \frac{a_1}{n^2 - \omega^2} \cos \omega t$$

The second auxiliary equation becomes

$$p^2\phi_2 + n^2\phi_2 = -h\phi_1{}^2$$

But ϕ_1 is now known from the previous solution. The repetition of this process gives the values of ϕ_2, ϕ_3, etc., with

$$x = c\phi_1 + c^2\phi_2 + \ldots \qquad . \quad . \quad (537)$$

This method will be used to obtain a solution for a variable inductive circuit.

Taylor's Series. In radio work non-linear circuits are the rule, and in dealing with these recourse is had to a mathematical expansion known as Taylor's series. This states that if $f(x)$ is a function of x such that the first n derivatives exist, and are finite and continuous in the interval from $x = a$ to $x = b$, then throughout the interval $f(x)$ may be expressed as

$$f(x) = f(x)_0 + (x - x_0)f'(x)_0 + \frac{(x - x_0)^2}{2!}f^2(x)_0 \ldots$$

$$+ \frac{(x - x_0)^{n-1}}{(n-1)!}f^{n-1}(x)_0 + R_n \quad . \quad (538)$$

where x, x_0 lie within the interval a, b, where all the derivatives are evaluated at $x = x_0$, and where R_n is a remainder, after n terms, which tends to zero in many (the useful) cases.

Thus, consider a rectifying device such that its EI characteristic is as shown in Fig. 49. Let the device operate close to E_0, I_0. Then, under steady operating conditions, a small variation of voltage to E will alter the current to

$$I = I_0 + K[E - E_0]$$

A closer approximation would be given by several terms—

$$I = I_0 + K_1[E - E_0] + K_2[E - E_0]^2 + \dots \quad (539)$$

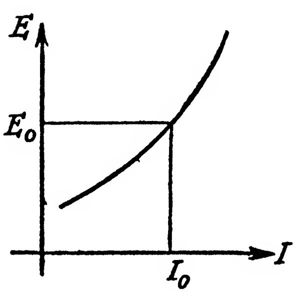

FIG. 49

To determine the coefficients K_1, $K_2 \dots$ we differentiate and put $E = E_0$. For the first differentiation we have

$$\frac{dI}{dE} = K_1 + 2K_2(E - E_0) \dots$$

$$\therefore \quad K_1 = \left[\frac{dI}{dE}\right] \text{at } E_0$$

The second yields

$$K_2 = \frac{1}{2!}\left[\frac{dI}{dE}\right] \text{at } E_0$$

and so we have

$$I = I_0 + \left[\frac{dI}{dE}\right]_{E_0}(E - E_0) + \left[\frac{d^2I}{dE^2}\right]_{E_0}\frac{(E - E_0)^2}{2!} + \dots \quad (540)$$

which, *in effect*, is Taylor's series.

With an applied voltage such as

$$E = E_0 + E_1 \sin \omega t + E_3 \sin 3\omega t$$

we get

$$I = I_0 + K_1[E_1 \sin \omega t + E_3 \sin 3\omega t] + K_2[E_1 \sin \omega t + E_3 \sin 3\omega t]^2$$

by taking two terms in the series (539), i.e.

$$I = I_0 + K_1[E_1 \sin \omega t + E_3 \sin 3\omega t] + K_2[E_1{}^2 - E_1{}^2 \cos 2\,\omega t$$
$$+ 2E_1E_3 \cos 2\omega t - 2E_1E_3 \cos 4\omega t + E_3{}^2 - E_3{}^2 \cos 4\omega t] \dots \quad (541)$$

This expression shows that in this case the current response consists of components

(i) due to E_0 acting alone.

(ii) due to terms of the frequency of the applied voltage,

(iii) due to terms whose frequency is the sum or difference of the applied frequencies.

Graphical methods of obtaining the response will be seen in texts dealing with radio engineering problems.

4. Some Formal Mathematical Methods. As illustrations of some of the mathematical methods that may be used in solving variable circuits we now consider simple circuits such as

(i) RL circuit with R constant and with inductance varying as $L = L_0(1 + \varepsilon \cos \omega t)$;

(ii) RL circuit with resistance given by $R = R_0(1 + a \cos \omega t)$ and L constant;

(iii) LC circuit with L constant and with capacity given by $\frac{1}{C} = \frac{1}{C_0}(1 + a \cos \omega t)$.

These elementary circuits will suffice to demonstrate the difficulties that the solutions entail.

(i) VARIABLE-INDUCTIVE CIRCUIT. The fundamental equation is now written as

$$Ri + pL_0(1 + \varepsilon \cos \omega t)i = E$$

Denoting ωt by x, we have $\omega dt = dx$ and with $\frac{R}{\omega L_0} = \alpha$, $e = E/\omega L_0$ we get

$$\left[\alpha + \frac{d}{dx}(1 + \varepsilon \cos x)\right] i = e \qquad . \qquad . \quad (542)$$

Assuming a solution, such as

$$i = i_0 + \varepsilon i_1 + \varepsilon^2 i_2 \ . \ . \ . = \sum_{n=0}^{n=\infty} \varepsilon^n i_n \qquad . \quad (543)$$

substituting it in equation (542), and equating like powers of ε, we get

$$\alpha i_0 = e, \quad \text{or} \quad i_0 = \frac{e}{\alpha}$$

Dealing with the first power of ε, we have

$$\alpha i_1 + \frac{di_1}{dx} + \frac{d}{dx}(i_0 \cos x) = 0$$

Hence
$$\left(\alpha + \frac{d}{dx}\right) i_1 = \frac{e}{\alpha} \cdot \sin x$$

Denoting $\frac{d}{dx}$ by D, we have, in operational notation—

$$i_1 = \frac{e}{\alpha} \cdot \frac{1}{D + \alpha} \cdot \frac{D}{D^2 + 1}$$

The solution for the *steady-state* condition is given by

$$i_1 = \frac{e}{\alpha(\alpha^2 + 1)} (\alpha \sin x - \cos x)$$

For the second power of ε we have

$$\alpha i_2 + D i_2 + D(i_1 \cos x) = 0$$

or
$$(\alpha + D) i_2 = \frac{e}{\alpha(\alpha^2 + 1)} [\alpha \cos 2x + \sin 2x]$$

$$\therefore \qquad i_2 = \frac{e}{\alpha(\alpha^2 + 1)} \cdot \frac{1}{\alpha + D} \left(\frac{\alpha D^2}{D^2 + 4} + \frac{2D}{D^2 + 4} \right)$$

The *steady-state* solution is given by

$$i_2 = \frac{e}{\alpha(\alpha^2 + 1)(\alpha^2 + 4)} \cdot [(\alpha^2 - 2) \cos 2x - 3\alpha \sin 2x]$$

In a similar fashion the values of i_3, i_4 . . . may be evaluated. So the full solution is given by

$$\begin{aligned} i &= \frac{e}{\alpha} \left\{ 1 + \frac{1}{\alpha^2 + 1} (\alpha \sin x - \cos x) \right. \\ &\qquad \left. + \frac{1}{\alpha^2 + 4} [(\alpha^2 - 2) \cos 2x - 3\alpha \sin 2x] + \ldots \right\} \\ &= \frac{E}{R} \left[1 + \frac{\alpha}{\alpha^2 + 1} \sin \omega t - \frac{3\alpha}{\alpha^2 + 4} \sin 2\omega t + \ldots \right. \\ &\qquad \left. - \frac{1}{\alpha^2 + 1} \cos \omega t + \frac{\alpha^2 - 2}{\alpha^2 + 4} \cos 2\omega t - \ldots \right] \quad . \quad (544) \end{aligned}$$

Thus it is seen that an inductance of this nature would give rise to several frequencies in the response.

(ii) Variable-resistance Circuit. The fundamental equation is now

$$Lpi + R(1 + a \cos \omega t) i = E$$

From the theory of differential equations the solution is obtained as follows.

Using the transformation $w = gi$, we have

$$\frac{dw}{dt} + i\left[\frac{Rg}{L}(1 + a \cos \omega t) - \frac{dg}{dt}\right] = \frac{Eg}{L} \quad . \quad (545)$$

We choose g such that

$$\frac{1}{g} \cdot \frac{dg}{dt} = \frac{R}{L}(1 + a \cos \omega t)$$

i.e.

$$g = \varepsilon^{\alpha t + \beta \sin \omega t}$$

where $\alpha = \dfrac{R}{L}$ and $\beta = \dfrac{Ra}{\omega L}$.

By substituting in (545) we have $dw/dt = Eg/L$.

or

$$w = \frac{E}{L}\int g dt = \frac{E}{L}\int \varepsilon^{\alpha t + \beta \sin \omega t} + A$$

where A is an integration constant.

So

$$i = \left[A + \frac{E}{L}\int \varepsilon^{\alpha t + \beta \sin \omega t}\right] \varepsilon^{-\alpha t}\, \varepsilon^{-\beta \sin \omega t} \quad . \quad (546)$$

From the theory of Bessel functions we have

$$\varepsilon^{\beta \sin \omega t} = J_0(\beta/j) + 2\sum_1^\infty \begin{Bmatrix} \cos n\omega t \ (n \text{ even}) \\ j \sin n\omega t \ (n \text{ odd}) \end{Bmatrix} J_n(\beta/j)$$

where it is noted that the Bessel is independent of time. So that

$$\int \varepsilon^{\alpha t + \beta \sin \omega t} dt = [J_0(\beta/j)] \cdot \frac{\varepsilon^{\alpha t}}{\alpha} + 2\sum_1^\infty \int \varepsilon^{\alpha t} \cdot \begin{Bmatrix} \cos n\omega t \\ j \sin n\omega t \end{Bmatrix} J_n(\beta/j) dt$$

Now $\displaystyle\int \varepsilon^{\alpha t} \begin{Bmatrix} \cos n\omega t \\ \sin n\omega t \end{Bmatrix} dt$

$$= \alpha\varepsilon^{\alpha t} \frac{[\varepsilon^{jn\omega t} + (-1)^n \varepsilon^{-jn\omega t}] - jn\omega\, [\varepsilon^{jn\omega t} + (-1)^n \varepsilon^{-jn\omega t}]}{\alpha^2 + n^2\omega^2}$$

When n is even, this reduces to

$$2\varepsilon^{\alpha t}\left[\frac{\alpha \cos n\omega t + n\omega \sin n\omega t}{\alpha^2 + n^2\omega^2}\right]$$

and when n is odd, to

$$2j\varepsilon^{\alpha t}\left[\frac{\alpha \sin n\omega t - n\omega \cos n\omega t}{\alpha^2 + n^2\omega^2}\right]$$

On denoting n by $2q$ and $2q + 1$, we have

$$\varepsilon^{\alpha t+\beta \sin \omega t} = \frac{\varepsilon^{\alpha t}}{\alpha} \cdot J_0 (\beta/J) + 2 \sum_1^\infty J_{2q} (\beta/j) \cdot \varepsilon^{\alpha t} \left[\frac{\alpha \cos 2q\omega t + 2q\omega \sin 2q\omega t}{\alpha^2 + 4q^2\omega^2} \right]$$

$$+ 2j \sum_{q=1}^{q=\infty} J_{2q+1} (\beta/j) \cdot \varepsilon^{\alpha t} \left[\frac{\alpha \sin (2q+1) \omega t - (2q+1) \omega \cos (2q+1) \omega t}{\alpha^2 + (2q + 1)^2 \omega^2} \right]$$

So the value of i becomes

$$= A\varepsilon^{-\alpha t-\beta \sin \omega t} + \frac{2E}{L} \cdot \varepsilon^{-\beta \sin \omega t} \left\{ \sum_1^\infty \frac{J_{2q} (\beta/j)}{\alpha^2 + 4q^2\omega^2} (\alpha \cos 2q\omega t + 2\, q\omega \sin 2\, q\omega t) \right.$$

$$+ j \sum_{q=1}^{q=\infty} \frac{J_{2q+1} (\beta/j)}{\alpha^2 + (2q+1)^2\omega^2} \cdot [\alpha \sin (2q+1) \omega t - (2q+1)\omega \cos (2q+1)\omega t]$$

$$\left. + \frac{1}{2\alpha} J_0 (\beta/j) \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (547)$$

From the expansion of the Bessel functions it will be seen that all terms are real, whether n is even or odd. The introduction of the variable resistance brings in a whole array of frequencies. From (i) and (ii) it is seen that the circuit deviates considerably from the invariable state.

(iii) Varying Capacity. We have already used this circuit as an illustration of the method of approximations. We revert to it here as giving us a form of Mathieu's equation. Here we have

$$Lpi + \frac{i}{pC} = E$$

But $\dfrac{1}{C} = \dfrac{u}{C_0} (1 + a \cos \omega t) = S_0 + S\,(t)$, say

where ω is the angular velocity of the cam shown in Fig. 48. Thus we have

$$Lp^2i + i\,[S_0 + S\,(t)\,] = 0 \quad . \quad . \quad (548)$$

This is a form of Mathieu's equation of which the explicit solution is unknown. However, from an electrical point of view we are able to discuss the circuit, and so the equation. If the variable term is absent, $S\,(t) = 0$, and it is known that the response is periodic with $\omega_0{}^2 = 1/LC$. When the variable

portion of the capacity is inserted, then we have ripples superimposed on the original periodic response. This combination gives a "beat" phenomenon. Let the period of recurrence be T, and suppose that at $t = 0$, $q = Q_0$ and $pq = I_0$. For the next period we may assume that $q = mQ_0$ and $pq = mI_0$, where m is some multiplier. If m is less than unity, the oscillations are decreasing and the circuit is stable; but if m is greater than unity, the oscillations are increasing and the circuit is unstable.

If we imagine the cam to give a rectangular ripple so that $S(t) = \pm 1$, then with $\omega_0^2 = 1/LC_0$ we may rewrite the equation as

$$p^2 i + \left(\omega_0^2 + \frac{\Delta K}{L}\right) i = 0 \qquad . \qquad . \quad (549)$$

This will hold for an interval, say $0 < \omega t < \pi$.
During the next interval $\pi < \omega t < 2\pi$ we have

$$p^2 i + \left(\omega_0^2 - \frac{\Delta K}{L}\right) i = 0 \qquad . \qquad . \quad (550)$$

These equations may be integrated as

$$i_1 = C_1 \sin q_1 t + C_2 \cos q_1 t \qquad . \qquad . \quad (551)$$

where $q_1 = \sqrt{\left(\omega_0^2 + \frac{\Delta K}{L}\right)}$

and

$$i_2 = C_3 \sin q_2 t + C_4 \cos q_2 t \qquad . \qquad . \quad (552)$$

where $q_2 = \sqrt{\left(\omega_0^2 - \frac{\Delta K}{L}\right)}$.

At $\omega t = \pi$ these solutions should fit as regards both amplitude and slope; at the end of a complete period they must be m times as large as at the beginning. So

$$(i_1)_{\omega t = \pi} = (i_2)_{\omega t = \pi}$$

$$\frac{di_1}{dt}_{\omega t = \pi} = \frac{di_2}{dt}_{\omega t = \pi}$$

$$(i_2)_{\omega t = 2\pi} = m\,(i_1)_{\omega t = 0}$$

$$\frac{di_1}{dt}_{\omega t = 2\pi} = m\,\frac{di_2}{dt}_{\omega t = 0}$$

On writing these equations in full we have, with $\pi/\omega = \theta$, that

$$C_1 \sin q_1\theta + C_2 \cos q_1\theta - C_3 \sin q_2\theta - C_4 \cos q_2\theta = 0$$

$$C_1 q_1 \cos q_1\theta - C_2 q_1 \sin q_1\theta - C_3 q_2 \cos q_2\theta + C_4 q_2 \sin q_2\theta = 0$$

$$0 + mC_2 - C_3 \sin 2q_2\theta - C_4 \cos 2q_2\theta = 0$$

$$-\,mC_1 q_1\,. + 0\,. - C_3 q_2 \cos 2q_2\theta + C_4 q_2 \sin 2q_2\theta = 0$$

These equations will give the solutions of the C_s if the determinant formed by omitting the C_s from the above is zero. This gives

$$m^2 - 2m\left(\cos q_1\theta \, . \cos q_2\theta - \frac{q_1{}^2 + q_2{}^2}{2q_1q_2}\sin q_1\theta \sin q_2\theta\right) + 1 = 0 \quad . \quad (553)$$

Thus $m = A \pm \sqrt{(A^2 - 1)}$, where A is the term within the brackets.

When $A > 1$, then $m > 1$, so that the system is unstable.
If $A = +1$, the system is seen to be stable.
When A is between $+1$ and -1, m is a complex number and does not agree with the original assumption. The real part is less than unity, so that a stable system could be expected.

With A less than -1, one of the values of m will be smaller than -1, i.e. the amplitude is reversed. The other will be greater than -1. So, after consecutive periods, we have

1st	2nd	3rd	4th	. . . periods
m	m^2	m^3	m^4	. . . amplitude multiplier
$-$	$+$	$-$	$+$	. . . sign of amplitude

It is seen that the amplitude of the third period will be in the same direction as the first, but will be larger. The criterion for stability is thus that

$$\cos q_1\theta \, . \cos q_2\theta - \frac{q_1{}^2 + q_2{}^2}{2q_1q_2} \sin q_1\theta \, . \sin q_2\theta \quad . \quad (554)$$

is numerically less than unity.*

5. **Linear Integral Types of Equation.** The form of the equation given in Section 1 is known as the linear integral type or Volterra type. To introduce equations of these types we may consider an equation such as

$$y_n + a_1(x) \, . \, y_{n-1} + a_2(x) \, . \, y_{n-2} \, . \, . \, . + a_n(x) \, . \, y = \phi(x) \quad (555)$$

Let $y_n = \dfrac{d^ny}{dx^n} = u(x)$, say.

* For a fuller discussion of this problem, see VAN DER POL AND STRUTT: *Phil. Mag.* (1928), Vol. 5, p. 18.

Then

$$y_{n-1} = \int u(x)\, dx + C_1$$

$$y_{n-2} = \int\int u(x)\,.\,dx + C_1 x + C_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y = \underbrace{\int \ldots \int}_{n \text{ times}} u(x)\,.\,dx + C_1 \frac{x^{n-1}}{(n-1)!} + C_2 \frac{x^{n-2}}{(n-2)!} \ldots + C_n$$

On substituting in equation (555), we get

$$u(x) + a_1(x) \int u(x)\, dx + a_2(x)\,.\int\int u(x)\, dx \ldots = \phi(x) + \Sigma C(x) = f(x), \quad \text{say}$$

where $\Sigma C(x) = a_1 C_1 + a_2 C_2 x + \ldots$

Now noting that $\int_0^x u(t)\,.\,dt^n = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} u(t)\,.\,dt$, we have

$$u(x) + \int_0^x \left(a_1(x) + a_2(x)\,.\,(x-t) + a_3(x)\,.\,\frac{(x-t)^2}{2!} \ldots \right) u(t) dt = f(x)$$

or

$$u(x) = f(x) - \int_0^x [a_1(x) \ \ldots] \, u(t)\, dt$$

$$= f(x) - \int_0^x K(x_1 t)\,.\,u(t)\,.\,dt$$

where $K(x_1 t) = [a_1(x) + a_2(x)\,.\,(x-t) \ \ldots]$; an equation which is of the same form as given under Section 1.

The specific forms of this type of equation are named after the mathematicians Abel, Poisson, Fredholm, and Volterra. Equations of this class are solved usually by methods of successive substitutions on the lines of Caqué's method of Chapter I. As the question of convergence arises, a full discussion is beyond our scope. For numerical work these solutions are practically useless in view of the slow convergency, and so step-by-step methods as discussed by Whittaker and Robinson* are used. There is one development of interest based on operational methods due to Koizumi.†

* WHITTAKER AND ROBINSON: *Calculus of Observations. Loc. cit.*
† KOIZUMI: *Phil. Mag.* (1931), Vol. 11, p. 432.

6. Solutions of Equations having a Kernel $K\ (x-\lambda)$ by Operational Methods. The Carson integral equation (Chapter X (7)) is given by

$$\frac{1}{pH(p)} = \int_0^\infty h(t)\,.\,\varepsilon^{-pt}\,dt$$

or in operational notation

$$\frac{1}{H(p)} \doteqdot h(t)$$

If $H(p)$ can be factorized in the form $H_1(p)\,.\,H_2(p)$, and if the auxiliary operational equations are

$$\frac{1}{H_1(p)} \doteqdot h_1(t) \text{ and } \frac{1}{H_2(p)} \doteqdot h_2(t),$$

then, by Borel's theorem—

$$h(t) = \frac{d}{dt}\int_0^t h_1(\lambda)\,.\,h_2\,(t-\lambda)\,d\lambda$$

So

$$\int_0^\infty h\,(t)\,\varepsilon^{-pt}\,dt \doteqdot \frac{1}{pH(p)} = \frac{p}{pH_1(p)\,.\,pH_2(p)}$$
$$= \int_0^\infty \left[\frac{d}{dt}\int_0^t h_1(\lambda)\,.\,h_2\,(t-\lambda)\,d\lambda\right]\varepsilon^{-pt}\,dt$$

On dealing with an equation of the Poisson type, such as

$$f(x) = \phi(x) - \alpha\int_0^x \phi(\lambda)\,.\,K\,(x-\lambda)\,d\lambda \qquad . \quad (556)$$

we multiply throughout by ε^{-Dx} and integrate with respect to x between 0 and ∞, obtaining

$$\int_0^\infty f(x).\varepsilon^{-Dx}dx = \int_0^\infty \phi(x).\varepsilon^{-Dx}.dx - \alpha\int_0^\infty\left[\int_0^x \phi(\lambda).K(x-\lambda).d\lambda\right]\varepsilon^{-Dx}.dx$$
$$. \quad . \quad . \quad . \quad . \quad . \quad (557)$$

Operationally we have

$$\frac{1}{D\phi(D)} = \int_0^\infty \phi(x)\,.\,\varepsilon^{-Dx}\,.\,dx = A(D), \quad \text{say}$$
$$\frac{1}{D\,.\,K(D)} = \int_0^\infty K(x)\,.\,\varepsilon^{-Dx}\,.\,dx = B(D)$$
$$\frac{1}{D\,.\,F(D)} = \int_0^\infty f(x)\varepsilon^{-Dx}\,.\,dx = C(D)$$

We have the following operational equation—

$$C(D) = A(D) - \alpha\left[\frac{A(D) \,.\, B(D)}{D}\right]. \qquad . \quad (558)$$

So
$$A(D) = \frac{C(D)}{1 - \alpha\dfrac{B(D)}{D}}$$

Now let
$$B(D) = \frac{R(D)}{\alpha\dfrac{R(D)}{D} - 1}$$

Then
$$A(D) = -\,C(D)\left[\alpha\frac{R(D)}{D} - 1\right]$$

$$= C(D) - \alpha\frac{R(D) \,.\, C(D)}{D} \qquad . \quad . \quad (559)$$

This may be converted to

$$\phi(x) = f(x) - \alpha\int_0^x f(\lambda) \,.\, L\,(x - \lambda)\, d\lambda \qquad . \quad (560)$$

where $L(x) \risingdotseq R(D)$.
Thus, suppose we have an equation such as

$$f(x) = \phi(x) - \alpha\int_0^x \phi(\lambda) \,.\, \sin\,(x - \lambda) \,.\, d\lambda \qquad . \quad (561)$$

Here $K(x) = \sin x$, or in operational notation

$$K(x) \risingdotseq \frac{D}{D^2 + 1} = B(D)$$

Now
$$R(D) = \frac{B(D)}{\dfrac{\alpha}{D} \,.\, B(D) - 1} = -\frac{D}{D^2 + (1 - \alpha)}$$

$$\risingdotseq -\frac{\sin\sqrt{(1 - \alpha)} \,.\, x}{\sqrt{(1 - \alpha)}}$$

$$= L(x).$$

So we now have

$$\phi(x) = f(x) + \alpha\int_0^x f(\xi)\,\frac{\sin\sqrt{(1 - \alpha)} \,.\, (x - \xi)}{\sqrt{(1 - \alpha)}}\, d\xi \quad (562)$$

When $f(x)$ is known, the value of $\phi(x)$ is obtained provided the integral is tractable. With an equation such as

$$\phi(t) = 1 + \int_0^x (\lambda - t) \,.\, \phi(\lambda) \,.\, d\lambda$$

$$K(t) = -t = -\frac{1}{p} = B(D)$$

$$R(p) = \frac{p}{p^2 + 1} \doteqdot \sin x$$

So $$\phi(t) = 1 + \int_0^x . \sin (x - \xi) d\xi = \cos t$$

Again, if $\phi(x) = \displaystyle\int_0^x \frac{f(\lambda)}{(x - \lambda)^\mu} d\lambda, \mu < 1.$ . . . (563)

Then, by Campbell and Foster's Table (No. 501), we have

$$D^{n-\alpha+1} \doteqdot \frac{1}{\Gamma(\alpha - n)} x^{\alpha - n - 1}$$

So that $$x^{\alpha-n-1} = \Gamma(\alpha - n) \,.\, D^{n-\alpha+1}$$

With $\alpha = 1$ and $n = \mu$, this reduces to

$$x^{-\mu} = \Gamma(1 - \mu) D^\mu = B(D)$$

Here $A(D) = \dfrac{C(D)}{\frac{B(D)}{D}} = \dfrac{C(D)}{\Gamma(1 - \mu) \,.\, D^{\mu - 1}} = \dfrac{C(D)}{\Gamma(1 - \mu)} \cdot \dfrac{x^{\mu - 1}}{\Gamma(\mu)}.$

But $\dfrac{1}{\Gamma(1 - \mu) \,.\, \Gamma(\mu)} = \dfrac{\sin \mu\pi}{\mu}$, and we now get

$$\phi(x) = \frac{\sin \mu\pi}{\mu} \int_0^x f(\lambda) \,.\, \frac{d\lambda}{(x - \lambda)^{1-\mu}}$$

7. **Graphical Methods.** Possibly for practical work, where the solution of a definite problem is required, graphical methods of integration are most used. From the results obtained no generalization can be made, and every new set of conditions requires a fresh start on the problem. We now illustrate the methods by considering such simple cases as the rise of voltage and of current in an iron-cored coil, in the field circuit of a dynamo, and in a circuit now known as the ferro-resonant circuit.

RISE OF VOLTAGE AND CURRENT IN A COIL. We consider next how the current in a coil builds up. The final value of the current at steady state will be V/R, where V is the applied voltage and R is the resistance. On neglecting leakage flux, we have that

$$V = Ri + N\frac{d\phi}{dt} \qquad . \quad . \quad (564)$$

where N is the number of turns on the coil.
Now, $V = RI$; so we have

$$R\,[I - i] = N\frac{d\phi}{dt}$$

We may write this as

$$R\,.\,\Delta i = N\frac{d\phi}{dt} \qquad . \quad . \quad . \quad (565)$$

Solving this equation, we get

$$t = \frac{N}{R}\int_{\phi_0}^{\phi_1}\frac{d\phi}{\Delta i} \qquad . \quad . \quad . \quad (566)$$

where ϕ_0 is the initial flux at $t = 0$, while ϕ_1, is the flux at any given time.

We now introduce a quantity $\Delta\phi$, which is a linear function of Δi, so that

$$\frac{\Delta\phi}{\Delta i} = \frac{\phi_s}{I} \qquad . \quad . \quad . \quad . \quad (567)$$

where ϕ_s and I are the steady-state values.
Using these values, we have

$$t = \frac{N\phi_s}{IR}\int_{\phi_0}^{\phi_1}\frac{d\phi}{\Delta\phi} \qquad . \quad . \quad . \quad (568)$$

Now, $N\phi_s = L_s I$, where L_s is the self-inductance of the coil at ϕ_s.

Then
$$t = \frac{L_s}{R}\cdot\int_{\phi_0}^{\phi_1}\frac{d\phi}{\Delta\phi} \qquad . \quad . \quad . \quad (569)$$

This equation may be solved by a graphical method due to Rüdenberg.*

For any coil we may obtain the flux current relationship as shown in Fig. 50, where are represented the quantities involved

* RÜDENBERG: *Elektrische Schaltvoränge* (Springer).

in the differential equation. We can now plot $(\phi, 1/\Delta\phi)$ direct. The value of the integral from $\phi_0 = 0$ to $\phi = \phi_1$ is given by the area $ABCD$. On denoting L_s/R by T, the circuit time constant, we now have that the area $ABCD = t/T$. Thus we know the value of t/T necessary to give a flux ϕ_1. Repetition of this process gives the curve $(\phi, t/T)$, and by reference to the magnetizing curve we are able to obtain the curve $(i, t/T)$.

RISE OF CURRENT IN THE FIELD CIRCUIT OF A DYNAMO. If ϕ be the air gap flux, then the generated voltage is

$$e = \phi Z\,(n/60)\,(p/a) \times 10^{-8} = K\phi, \text{ say.}$$

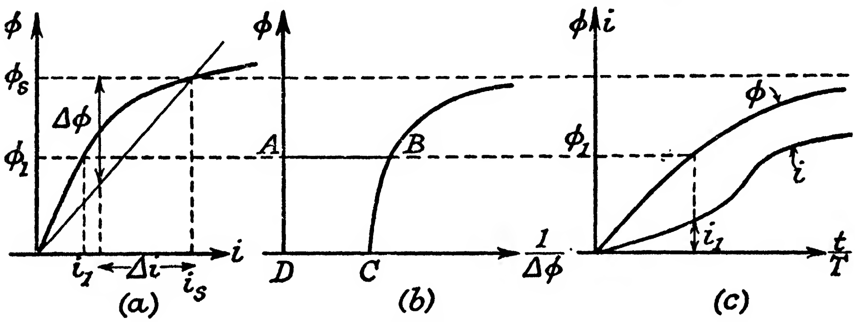

FIG. 50

The field flux is $\nu\phi$, where $(\nu - 1)$ represents a leakage factor, and so the voltage induced in the field windings by this flux is given by

$$\nu N \frac{d\phi}{dt} = \frac{\nu N}{K} \cdot \frac{de}{dt} \qquad . \quad . \quad . \quad (570)$$

so that the equation for a self-excited machine is

$$\frac{\nu N}{K} \cdot \frac{de}{dt} + Ri = e$$

or

$$\frac{\nu N}{K} \frac{de}{dt} = e - Ri = \Delta e \quad . \quad . \quad . \quad (571)$$

where Δe represents the difference between the generated volts and the resistance drop in the field circuit. The solution of the equation is

$$t = \frac{\nu N}{K} \int_{e_0}^{e_1} \frac{de}{\Delta e}$$

The time constant of the winding is given by

$$T = L_s/R = \frac{\nu N \phi_s}{Ri_s} = \frac{\nu N}{K} \qquad . \quad . \quad (572)$$

so that the simplest form of this equation is

$$t/T = \int_{e_0}^{e_1} (1/\Delta e)\, de \quad . \quad . \quad . \quad (573)$$

The magnetization curve (e, i) may be plotted from test results, and is as shown in Fig. 51 (a). All quantities in the above discussion are shown on this figure. The plot of $(1/\Delta e)$ against e is shown in Fig. 51 (b). The area $ABCD$ gives the right-hand side of equation (573). Thus we know the value of

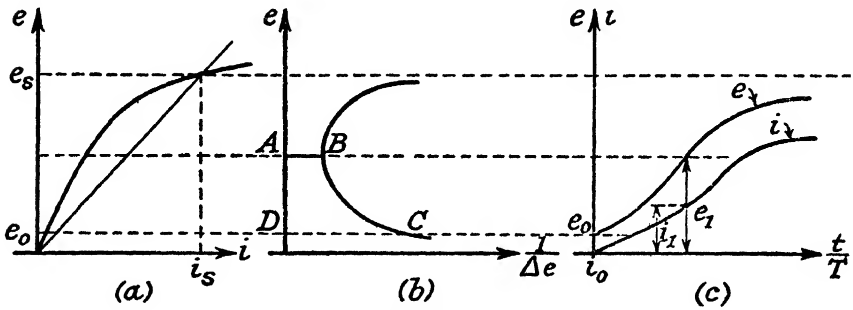

FIG. 51

t/T necessary for the voltage to rise to e_1. Repetition gives the characteristic $(e, t/T)$, and the $(i, t/T)$ function may be obtained by reference to the magnetization curve.

THE FERRO-RESONANT CIRCUIT: STEADY STATE. When an iron-cored coil is in series with a condenser, the voltage across the true inductance is given by $v(t) = L(i) \,.\, i(t)$. The form of this curve would be that of the magnetizing curve. Thus the expression for $v(t)$ would introduce insuperable difficulties if we were to proceed as outlined in Section 1. We will suppose that the iron-cored coil and the condenser are both available, and that test results connecting V and i are also available. To introduce the method we consider initially an LC circuit having constant parameters. The equation is

$$Lpi + (1/pC)i = E \sin \omega t$$

Here the potency of p is $j\omega$, and so $1/p$ is $1/j\omega$. Since the equation must hold for all values of time, we may rewrite it as

$$j(\omega LI - I/\omega C) = E_0 \quad . \quad . \quad (574)$$

or APPLIED VOLTS = INDUCTIVE DROP — INCAPACITY DROP

Here, the relationship between the inductive drop and the current is a straight line through the origin at an angle ϕ to

the current axis, so that $\tan \phi = \omega L$, as shown in Fig. 52. Now we select a point A on the voltage axis so that $OA = E_0$, and draw a line AB at an angle of θ degrees to the horizontal such that $\tan\ \theta = e_c/I = 1/\omega C$. It is seen now that OB_1 satisfies the equation $V_L - V_C = E$. So, for an applied voltage

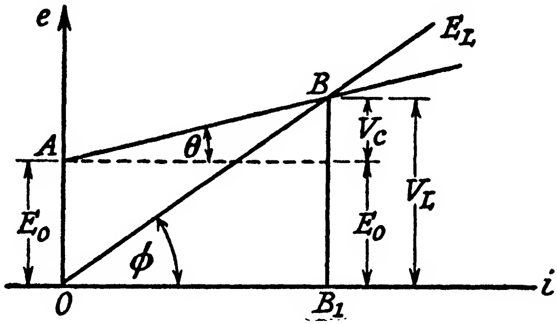

Fig. 52

OA, the current will be given by OB_1. Repetition of this construction will give the relationship between the current and the applied voltage.

When the frequency is varied, it is evident that ϕ will increase, while θ will decrease, with increase of frequency.

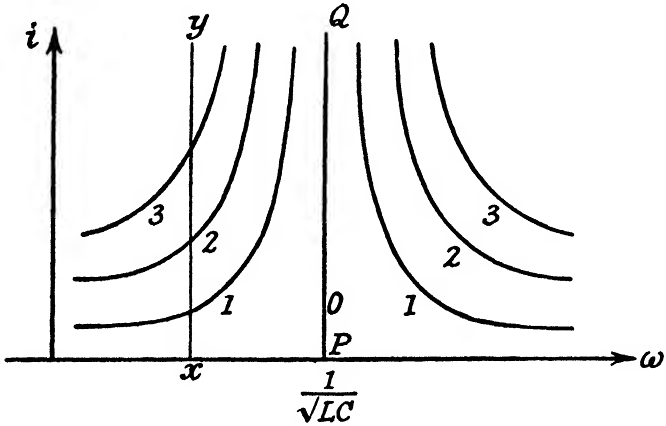

Fig. 53

So that at some frequency the lines AB and OB will be parallel with each other, viz. when $\omega^2 = 1/LC$.

If we plot the relationship (i, ω) for different values of the applied voltage, a set of curves such as those shown in Fig. 53 result for different values of E_0. Here values of E_0 are shown,

to some arbitrary scale, as 1, 2, 3 . . . . The curve marked 0 is the natural frequency curve. The current/voltage relationship at constant frequency is obtained if we select any frequency, such as xy, and plot the current against voltage. Then, should the voltage vary with time, the time and voltage are correlated and the current time relationship so obtained.

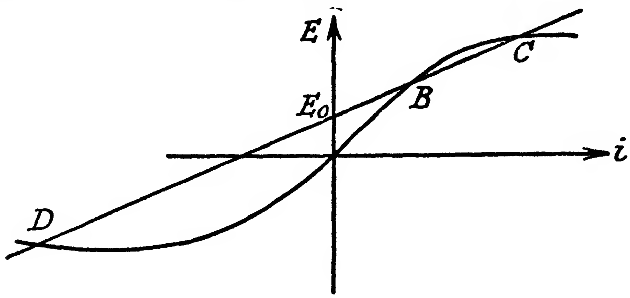

FIG. 54

Now, when we consider the iron-cored coil, we get the saturation characteristic of Fig. 54, instead of the straight line OB of Fig. 52. It will be observed that the capacity line, inserted at the correct applied voltage, cuts the inductor characteristic at the three points C, B, D. In the same way as for

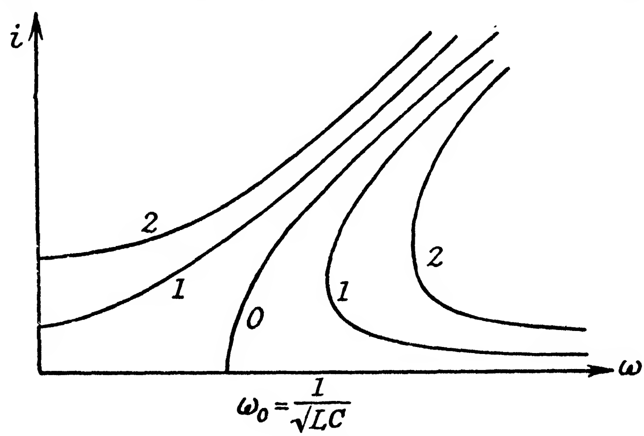

FIG. 55

the linear circuit, the current/frequency curve can be obtained for different values of the applied voltage (Fig. 55). The curve 0 here represents the natural frequency curve. It will be observed that in Fig. 54 the locus of the point D gives the curve to the left of $\omega = \omega_o$, while B gives the lower and C gives

the upper part of the right-hand half of the curve. From Fig. 55 it will be observed that a decreasing voltage at constant frequency gives an increased current, which is contrary to established principles. Thus the upper part of the right-hand curve is unstable. All the others are stable. When the voltage applied to this circuit is gradually increased, it will be observed that, at a certain voltage, the current suddenly increases from a very low value corresponding to the point B of Fig. 54 to a high value corresponding to D when partial resonance conditions exist.

8. **Empirical Formulae.** When an empirical formula is available, the circuit equation may be solved mathematically. However, any empirical formula, at best, is an approximation, and this fact must not be forgotten.

BUILDING-UP OF VOLTAGE OF SHUNT DYNAMO. To illustrate this method we will consider how the voltage builds up on a shunt dynamo, running at constant speed. We consider the field to have a constant inductance L, and we neglect the leakage flux. The e.m.f. across the brushes is

$$E = Ri + pLi = Ri + np\phi \times 10^{-2} \qquad . \quad (575)$$

where the flux is expressed in megalines and n is the number of turns in the shunt field. Froelich's equation may be written as $i = \phi/(a - b\phi)$, and the voltage across the brushes is given by $E = \phi M$ or

$$\phi M = \frac{R\phi}{a - b\phi} + n\frac{d\phi}{dt} \times 10^{-2}$$

so that

$$\frac{100}{n}\,dt = \frac{a - b\phi}{\phi[(aM - R) - bM\phi]}\,d\phi$$

$$= \left[\frac{a}{\phi(aM - R)} + \frac{bR}{(aM - R)\,(Ma - R - bM\phi)}\right]d\phi$$

Integrating both sides, we have that

$$\frac{100t}{n} = \left[\frac{a}{aM-R}\cdot \log_e\phi - \frac{R}{M(aM-R)}\cdot \log_e(Ma-R-bM\phi)\right]_0^t$$

At $t = 0$ let $\phi = \phi_R$, the residual flux. Making this substitution, we have

$$\frac{100t}{n} = \frac{a}{aM-R}\cdot \log_e\frac{\phi}{\phi_R} - \frac{R}{M(aM-R)}\cdot \log_e\left(\frac{aM-R-bM\phi}{aM-R-bM\phi_R}\right)$$

Denoting $M\phi$ by E and $M\phi_R$ by E_R, we get

$$\frac{100t}{n} = \frac{a}{aM-R} . \log_e \frac{E}{E_R} - \frac{R}{M(aM-R)} . \log_e \left(\frac{aM-R-E}{aM-R-E_R} \right) . \quad (576)$$

This expression gives the relationship between E and t.

9. **Maintenance of Oscillations.** In radio work, oscillations are maintained in a tuned RLC circuit by the addition of a thermionic valve. The explanation that the valve adds a negative resistance to the circuit means that the total circuit resistance must be zero precisely. However, such a delicate condition is not required in practice, and so we conclude that the linear theory is inadequate. Rayleigh met the same difficulty.* For a linear RLC circuit we have

$$p^2x + apx + \omega^2 x = 0$$

and for a non-linear circuit we get an equation, such as

$$p^2x + \omega f(x) . px + \omega^2 x = 0$$

If $\dot{y} = px$, then

$$p^2x = \frac{d\dot{y}}{dx} \cdot \frac{dx}{dt} = \dot{y}\frac{d\dot{y}}{dx}$$

So we get $$\dot{y}\frac{d\dot{y}}{dx} + \omega f(x) . \dot{y} + \omega^2 x = 0$$

i.e. $$\frac{\dot{y}}{\omega}\frac{d\dot{y}}{dx} + f(x)y + \omega x = 0$$

Let $q = \dfrac{\dot{y}}{\omega} + F(x)$, where $F(x) = \int_0^x f(x)dx$.

The equation then becomes

$$dq/x = dx/[F(x) - q]$$

Poincaré† and Lienard‡ have studied this form of equation, and it seems, from their work, that the explicit solution will probably remain unknown.

Before considering the graphical method we will consider an equation of this type from a mathematical point of view. To illustrate, we will consider a simple oscillator circuit, which in its simplest form can be represented as in Fig. 56. This

* Rayleigh: *Theory of Sound*, Vol. I, Section 68*a*.

† Poincaré: *J. des Maths* (1882), Vol. 8, p. 251.

‡ Lienard: *R.G.E.* (1928), Vol. 23, pp. 901 and 946.

treatment is based on a paper by Van der Pol.* It will be assumed that the characteristic of the unstable circuit element is defined by a single-valued function of the independent variable, and that this characteristic is a symmetrical cubic

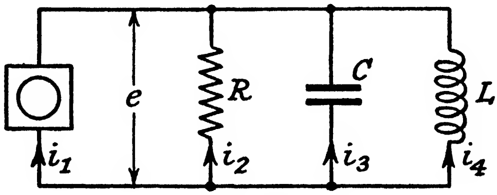

FIG. 56

function. So, for the circuit, we have the following relationships—

$$e = Ri_2 = Lpi_4 = (1/pC)i_3$$

$$i_1 = -\gamma(e - \beta e^3)$$

Also
$$i_1 + i_2 + i_3 + i_4 = 0 \qquad . \quad . \quad (577)$$

Hence the general equation is given as

$$-\gamma(e - \beta e^3) + Ge + \frac{1}{L}\int edt + C\frac{de}{dt} = 0 \qquad . \quad (578)$$

On differentiating again, we have

$$Cp^2e + [G - \gamma(1 - 3\beta e^2)]pe + e/L = 0 \qquad . \quad (579)$$

Using $x = t/\sqrt{(LC)} = \omega_0 t$, $\sigma = (\gamma - G)\sqrt{(L/C)}$, and $y = e\sqrt{[3\beta\gamma/(\gamma - G)]}$, we obtain that

$$\frac{d^2y}{dx^2} - \sigma(1 - y^2)\frac{dy}{dx} + y = 0 \qquad . \quad (580)$$

If $\sigma = 0$, then y is sinusoidal, with a period $= 2\pi/\omega_0$. As σ increases from 0, the oscillations deviate from the simple harmonic type. If σ is negative, the oscillations will be positively damped; but if positive, the amplitude of the oscillations will increase.

Let σ and $\sqrt{[3\beta\gamma/(\gamma - G)]}$ be small, so that we may assume that the oscillations are approximately sinusoidal and are given by $y = u \sin x$, where u is a slowly varying function of x. By differentiation we have

$$\frac{dy}{dx} = u \cos x + \sin x \, . \, \frac{du}{dx}$$

* VAN DER POL: *Proc. I.R.E.* (1934), Vol. 22, p. 1057.

Due to assumptions respecting σ, we will assume that $\sigma \dfrac{du}{dx}$ is negligibly small.

Again $\dfrac{d^2y}{dx^2} = 2\dfrac{du}{dx}\cos x - u \sin x$

Also
$$y^2\frac{dy}{dx} = \frac{1}{3}\cdot\frac{d(y^3)}{dx}$$
$$\cong \frac{u^3}{4}\cos x$$

So that equation (580) may be written
$$2u\frac{du}{dx} - \sigma u^2(1 - u^2/4) = 0 \quad . \quad . \quad . \quad (581)$$

Using the transformation $wu^2 = 1$, we have
$$dw/dx + \sigma(w - \tfrac{1}{4}) = 0$$

The solution of this equation is given as
$$w = \tfrac{1}{4}[1 + \varepsilon^{-\sigma(t-t_0)}] \quad . \quad (582)$$

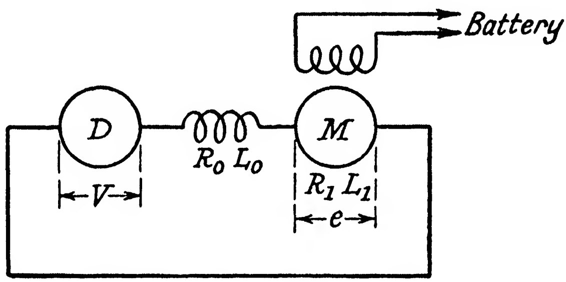

FIG. 57

where t_0 is a constant of integration. Then
$$u = 2[1 + \varepsilon^{-\sigma(t-t_0)}]^{-\frac{1}{2}}$$
$$y = u \sin x$$
$$\therefore\ e = 2\sqrt{\left(\frac{\gamma - G}{3\beta\gamma}\right)} \times \frac{\sin \omega_0 t}{[1 + \varepsilon^{-\sigma(t-t_0)}]^{\frac{1}{2}}} \quad . \quad (583)$$

After a very long time the amplitude of e will increase to
$$2\sqrt{[(\gamma - G)/3\beta\gamma]} \cong 2\sqrt{(1/3\beta)}$$
If we cannot make the saving assumption that σ is small, we must proceed by graphical means. To demonstrate, we will consider the case of a series dynamo supplying power to a separately excited shunt motor (Fig. 57). The armature of

the latter merely oscillates. The voltage equation becomes

$$V = iR_0 + L_0pi + R_1i + L_1pi + e$$

and for increasing speed we have $J\dfrac{d\omega}{dt} = K\phi i$ and $e = a\phi\omega$, where a and K are constants.

Let $R_0 + R_1 = R$ and $L_0 + L_1 = L$. Then we have

$$V = iR + Lpi + a\phi\omega = iR + Lpi + \frac{aK}{J}\phi^2\int idt \quad (584)$$

For a series dynamo the connection between the voltage and current is

$$V = \rho i - \tfrac{1}{3}\gamma i^3$$

where ρ and γ are constants.

The equation becomes

$$Ri + Lpi + \frac{aK}{J}\phi^2\int idt - (\rho - \tfrac{1}{3}\gamma i^2)i = 0$$

or $$Lp^2q + Rpq + \frac{aK}{J}\phi^2 q - [\rho - \tfrac{1}{3}\gamma(pq)^2]pq = 0 \quad . \quad (585)$$

Let $t = x\sqrt{LC}$. In other words, we are measuring time in (periods/2π). Then

$$p^2q = \frac{d}{dt}\left(\frac{dq}{dx}\cdot\frac{dx}{dt}\right) = \frac{d^2q}{dx^2}\cdot\left(\frac{dx}{dt}\right)^2 = \frac{1}{LC}\frac{d^2q}{dx^2}$$

Writing $\dfrac{aK}{J}\phi^2 = \dfrac{1}{C}$, we get

$$\frac{1}{C}\cdot\frac{d^2q}{dx^2} + \frac{R-\rho}{\sqrt{(LC)}}\cdot\frac{dq}{dx} + \tfrac{1}{3}\gamma\left(\frac{1}{LC}\right)^{\frac{3}{2}}\cdot\left(\frac{dq}{dx}\right)^3 + \frac{q}{C} = 0 \quad . \quad (586)$$

With $q = y\sqrt{[(R-\rho)LC/\gamma]}$, we have

$$\ddot{y} - \sigma(\dot{y} - \tfrac{1}{3}\dot{y}^3) + y = 0$$

where $$\sigma = \sqrt{[C(\rho - R)/L]}$$

If we write $Z = py$, $pZ = Z\dfrac{dZ}{dy}$, we get

$$Z\frac{dZ}{dy} - \sigma(Z - \tfrac{1}{3}Z^3) + y = 0 \quad . \quad (587)$$

The graphical solution is obtained in the following way. First, we construct the curve $y = \sigma[Z - \frac{1}{3}Z^3]$ as in Fig. 58. At any point M, MPQ is drawn parallel with the y axis. At the point P, where MPQ cuts the curve, PN is drawn perpendicular to Ox, and NM is drawn. At the selected point M we

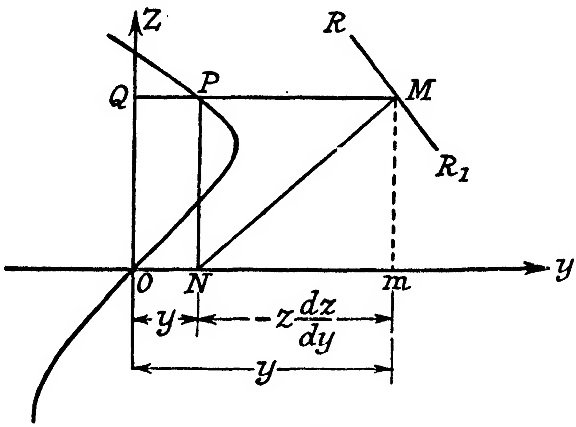

FIG. 58

draw the ordinate Mm. Now the subnormal through M, or $-Z\dfrac{dZ}{dy}$ is Nm, so that equation (587) will now read

$$Om = ON + Nm$$

A line RR_1 at right angles to MN will be a tangent to the required curve. Repetition of this process will give the locus of M as a family of spirals. Eventually, the volutes curl asympotically against a certain curve. This curve is a cyclogram, and by usual methods can be converted to a wave. Using the conversion factors, we revert to an (i, t) wave. This method is of the "hit-or-miss" variety: a judicious selection of the point M will assist in shortening the work. Hak* has given guides to the selection of initial values, based on the following development. Equation (585) may be written as

$$Lp^2i + (1/C)i + (R - \rho + \tfrac{1}{3}\gamma i^2)pi = 0$$

On using dimensionless time, we have

$$p^2i + \sqrt{\left(\frac{C}{L}\right)} \cdot (R - \rho)\left[1 + \tfrac{1}{3}\gamma \cdot \frac{1}{LC} \cdot \frac{1}{R - \rho} i^2\right] pi + i = 0$$

* HAK: *R.G.E.* (1935), Vol. 26, p. 895.

With $i = y\sqrt{[3LC(R - \rho)/\gamma]}$, we get

$$p^2y - \sigma(1 - y^2)py + y = 0 \qquad (588)$$

where $\sigma = (\rho - R)\sqrt{(C/L)}$.

At $t = 0$, $y = 0$, and then $dy/dt = C_1$: at any other time (y_n, t_n) the tangent is given by

$$\left(\frac{dy}{dt}\right)_{t_n} = C_1 - \int_0^t ydt + \sigma[y_n - \tfrac{1}{3}y_n{}^3] \qquad (589)$$

As the equation is valid for all values of time, a curve fulfilling this equation may be plotted by the usual step-by-step methods and the integral evaluated. Hak gives the following approximations for the initial values—

Slope: $C_1 = 2 + 0{\cdot}16\sigma^2$ (for $\sigma < 1$) . . . (590)

$= 1{\cdot}026 + 0{\cdot}654\sigma + 0{\cdot}489/\sigma$ (for $\sigma > 1$) (591)

Period: $T = \pi + 0{\cdot}167\sigma^2$ (for $\sigma < 1$) . . . (592)

$= 1{\cdot}989 + 0{\cdot}745\sigma + 0{\cdot}586/\sigma$ (for $\sigma > 1$) (593)

Max. value at t_{max}:

$$t_{max} = \frac{5{\cdot}728}{3{\cdot}365 + \sigma} - 0{\cdot}132 \qquad (594)$$

APPENDIX I

EQUIVALENCIES OF OPERATORS

Operator for $H_{(p)}$	Developed for y
$p/(p+\beta)$	$\varepsilon^{-\beta t}$
$p/(p+\beta)^n$	$[t^{n-1}/(n-1)!]\,\varepsilon^{-\beta t}$
$\beta/(p+\beta)$	$\varepsilon^{-\alpha t}-1$
With $\omega_0^2 > \alpha^2$; $\omega^2 = \omega_0^2 - \alpha^2$; $\tan\phi = \omega/\alpha$	
$p^2/(p^2+2\alpha p+\omega_0^2)$	$-(\omega_0/\omega)\,\varepsilon^{-\alpha t}\sin(\omega t-\phi)$
$p/(p^2+2\alpha p+\omega_0^2)$	$\varepsilon^{-\alpha t}(\sin\omega t)/\omega$
$1/(p^2+2\alpha p+\omega_0^2)$	$(1/\omega_0^2)\,[1-(\omega_0/\omega)\,\varepsilon^{-\alpha t}\sin(\omega t+\phi)]$
With $\alpha^2 > \omega^2$ (roots $-m$ & $-n$)	
$p^2/(p^2+2\alpha p+\omega_0^2)$	$[n\varepsilon^{-nt}-m\varepsilon^{-nt}]/(n-m)$
$p/(p^2+2\alpha p+\omega_0^2)$	$[\varepsilon^{-mt}-\varepsilon^{-nt}]/(n-m)$
$1/(p^2+2\alpha p+\omega_0^2)$	$(1/\omega_0^2)-[(\varepsilon^{-mt}/m)-(\varepsilon^{-nt}/n)]/(n-m)$
With $\alpha^2 = \omega_0^2$ (roots equal)	
$p^2/(p^2+2\alpha p+\omega_0^2)$	$\varepsilon^{-\alpha t}(1-\alpha t)$
$p/(p^2+2\alpha p+\omega_0^2)$	$t\varepsilon^{-\alpha t}$
$1/(p^2+2\alpha p+\omega_0^2)$	$[1-\varepsilon^{-\alpha t}(1+\alpha t)]/\omega_0^2$
$p^2/(p+\beta)$	$h(0)-\beta\varepsilon^{-\beta t}$
$p/(p+\beta)^2$	$t\varepsilon^{-\beta t}$
$p/(p-\beta)^2$	$-t\varepsilon^{\beta t}$
$p/(p+\alpha)(p+\beta)$	$(\varepsilon^{-\beta t}-\varepsilon^{-\alpha t})/(\alpha-\beta)$
$p^2/(p+\alpha)(p+\beta)$	$(\alpha\varepsilon^{-\alpha t}-\beta\varepsilon^{-\beta t})/(\alpha-\beta)$
$[\alpha\beta/(p-\alpha)(p+\beta)]-1$	$(\beta\varepsilon^{-\alpha t}-\alpha\varepsilon^{-\beta t})/(\alpha-\beta)$
$\dfrac{p}{(p+\alpha)(p+\beta)(p+\gamma)}$	$\dfrac{(\gamma-\beta)\varepsilon^{-\alpha t}+(\alpha-\gamma)\varepsilon^{-\beta t}+(\beta-\alpha)\varepsilon^{-\gamma t}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$
$\dfrac{p^2}{(p+\alpha)(p+\beta)(p+\gamma)}$	$\dfrac{\alpha(\beta-\gamma)\varepsilon^{-\alpha t}+\beta(\gamma-\alpha)\varepsilon^{-\beta t}+\gamma(\alpha-\beta)\varepsilon^{-\gamma t}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$
$\alpha p/(p^2+\alpha^2)$	$\sin\alpha t$

Operator for $H_{(p)}$	Developed for y
$p^2/(p^2+\alpha^2)$	$\cos \alpha t$
$\alpha^2 p/p\,(p^2+\alpha^2)$	$2\sin^2(\frac{1}{2}\alpha t)$
$p/(p+\beta+j\alpha)(p+\beta-j\alpha)$	$(\sin \alpha t\,.\,\varepsilon^{-\beta t})/\alpha$
$p/(p-\beta+j\alpha)(p-\beta-j\alpha)$	$-(\sin \alpha t\,.\,\varepsilon^{\beta t})/\alpha$
$p^2/(p+\beta+j\alpha)(p+\beta-j\alpha)$	$[\cos \alpha t-(\beta/\alpha)\sin \alpha t]\,\varepsilon^{-\beta t}$
$p^2/(p-\beta+j\alpha)(p-\beta-j\alpha)$	$-[\cos \alpha t+(\beta/\alpha)\sin \alpha t]\,\varepsilon^{\beta t}$
$p^2/(p^2-\omega^2)$	$\cosh \omega t$
$\omega p/(p^2-\omega^2)$	$\sinh \omega t$
$p^{\frac{1}{2}}\varepsilon^{1/4p}$	$(\cosh \sqrt{t}\,.)/\sqrt{(\pi t)}$
$[p/\sqrt{(p+\beta)}]\,\varepsilon^{\rho/(p+\beta)}$	$[\varepsilon^{-\beta t}\sqrt{(\pi t)}]\cosh 2\sqrt{(\rho t)}$
$[p/\sqrt{(p+\beta)^3}]\,\varepsilon^{\rho/(p+\beta)}$	$[\varepsilon^{-\beta t}\sqrt{(\pi t)}]\sinh 2\sqrt{(\rho t)}$
$1/p^n = p^{-n}$	$t^n/n!$
$p^{n-a+1} = p^{-(a-n-1)}$	$t^{a-n-1}/\Gamma(a-n)$
p^{-a+1}	$t^{a-1}/\Gamma(a)$
$p^{\frac{1}{2}}$	$t^{-\frac{1}{2}}/\Gamma(\frac{1}{2}) = 1/\sqrt{(\pi t)}$
$p^{\frac{1}{2}}f(t)$	$\dfrac{1}{\sqrt{\pi}}\dfrac{d}{dt}\displaystyle\int_0^t \dfrac{f(\lambda)d\lambda}{\sqrt{(t-\lambda)}}$
$p^{-\frac{1}{2}}$	$t^{\frac{1}{2}}/\Gamma(\frac{3}{2}) = 2\sqrt{(t/\pi)}$
$p^{\frac{3}{2}}$	$-\frac{1}{2}\pi^{-\frac{1}{2}}\,t^{-\frac{3}{2}}$
$p^{\frac{5}{2}}$	$\frac{3}{4}\pi^{-\frac{1}{2}}\,t^{-\frac{5}{2}}$
$p^{n-\frac{1}{2}+1}$	$(-1)^n.\dfrac{1.3.5\ldots(2n-1)}{(\frac{1}{2}\pi)^{\frac{1}{2}}}.(2t)^{-n-\frac{1}{2}}$
$p^{-n-\frac{1}{2}+1}$	$\dfrac{(\frac{1}{2}\pi)^{-\frac{1}{2}}}{1.3.5\ldots(2n-1)}(2t)^{n-\frac{1}{2}}$
$p(p+\beta)^{\frac{1}{2}}$	$-\frac{1}{2}\pi^{-\frac{1}{2}}\,\varepsilon^{-\beta t}\,t^{-\frac{3}{2}}$
$p(p+\beta)^{-\frac{1}{2}}$	$\varepsilon^{-\beta t}/\sqrt{(\pi t)}$
$p(p-\beta)^{-\frac{1}{2}}$	$\varepsilon^{\beta t}/\sqrt{(\pi t)}$ (if $t<0$)
$p(p+\beta)^{-\frac{3}{2}}$	$2\varepsilon^{-t}\sqrt{(t/\pi)}$ (if $t>0$)
$p(p+\beta)^{-\alpha}$	$\varepsilon^{-j2\pi\alpha(v+w)}\varepsilon^{-\beta t}\,t^{\alpha-1}/\Gamma(\alpha)$
$p(p-\beta)^{-\alpha}$	$-\varepsilon^{-j2\pi\alpha(v+w)}\,\varepsilon^{\beta t}\,t^{\alpha-1}/\Gamma(\alpha)$
$p[(p+\beta)^{1-\alpha}-(p+\gamma)^{1-\alpha}]$	$t^{\alpha-2}(\varepsilon^{-\beta t}-\varepsilon^{-\gamma t})/\Gamma(\alpha-1)$ if $t>0$

Operator for $H_{(p)}$	Developed for y
	With $p = h^2q^2$
$\varepsilon^{-q\alpha}$	$1 - \operatorname{erf}\left(\frac{\alpha}{2h\sqrt{t}}\right) = 1 - \sqrt{\left(\frac{\alpha}{\pi}\right)}\left[\frac{1}{t^{\frac{1}{2}}} - \frac{1}{2}\frac{\alpha}{3!}\frac{1}{t^{\frac{3}{2}}} + \frac{1}{2}\frac{3}{2}\frac{\alpha^2}{5!}\frac{1}{t^{\frac{5}{2}}} + \dots\right]$
$q\varepsilon^{-q\alpha}$	$[1/h\sqrt{(\pi t)}]\,\varepsilon^{-\alpha^2/4h^2t}$
$\varepsilon^{-q\alpha}/q$	$2h\sqrt{\left(\frac{t}{\pi}\right)}\,\varepsilon^{-\alpha^2/4h^2t} - x\left[1 - \text{erf.}\left(\frac{\alpha}{2h\sqrt{t}}\right)\right]$
$\left(\frac{a}{q+a}\right)\varepsilon^{-q\alpha}$	$1 - \operatorname{erf}\left(\frac{\alpha}{2h\sqrt{t}}\right) - \exp.\,(a^2h^2t + a\alpha)\left[1 - \operatorname{erf}\left(\frac{\alpha}{2h\sqrt{t}} + ah\sqrt{t}\right)\right]$
$\frac{p^{\frac{1}{2}}}{p+\gamma}$	$\frac{1}{\sqrt{(-\gamma)}}\,\varepsilon^{-\gamma t}\operatorname{erf}[\sqrt{(-\gamma t)}] \quad t > 0$ $= \frac{1}{\gamma\sqrt{(\pi t)}}\left[1 + \frac{1}{2\gamma t} + \frac{1\,.\,3}{(2\gamma t)^2} + \frac{1\,.\,3\,.\,5}{(2\gamma t)^3} + \dots\right]$
$\frac{p^{\frac{3}{2}}}{p+\gamma}$	$= \frac{1}{\sqrt{(\pi t)}} + \sqrt{(-\gamma)}\,\varepsilon^{-\gamma t}\operatorname{erf}[\sqrt{(-\gamma t)}] \quad t > 0$ $= -\frac{1}{\sqrt{(\pi t)}}\left[\frac{1}{2\gamma t} + \frac{1\,.\,3}{(2\gamma t)^2} + \frac{1\,.\,3\,.\,5}{(2\gamma t)^3} + \dots\right]$
$\frac{p}{(p+\gamma)\sqrt{(p+\beta)}}$	$\frac{1}{\sqrt{(\beta-\gamma)}}\,\varepsilon^{-\gamma t}\operatorname{erf}\sqrt{[(\beta-\gamma)t]}$
$\frac{p\sqrt{(p+\beta)}}{p+\gamma}$	$\frac{1}{\sqrt{(\pi t)}}\,\varepsilon^{-\beta t} + \sqrt{(\beta-\gamma)}\,\varepsilon^{-\gamma t}\operatorname{erf}\sqrt{[(\beta-\gamma)t]}$
$\frac{p^{\frac{3}{2}}}{1+\sqrt{(\beta p)}}$	$\frac{1}{\sqrt{\beta}}h(0) - \frac{1}{\beta\sqrt{(\pi t)}} + \frac{1}{\beta\sqrt{\beta}}\,\varepsilon^{t/\beta}\operatorname{erf}\sqrt{(t/\beta)}$
$\frac{\sqrt{\beta}}{\sqrt{(p+\beta)}} - 1$	erfc. $\sqrt{(\beta t)}$
$\sqrt{[(p/\beta) - 1]} - 1$	$\frac{1}{\sqrt{(\pi\beta t)}}\,\varepsilon^{-\beta t} - \text{erfc.}\sqrt{(\beta t)}$
$\frac{p}{1+\sqrt{(\beta p)}}$	$-\frac{1}{\beta}h(0) + \frac{2t-\beta}{2\beta t\sqrt{(\pi\beta t)}} - \frac{1}{\beta^2}\,\varepsilon^{t/\beta}\operatorname{erfc}\sqrt{(t/\beta)}$

Operator for $H_{(p)}$	Developed for y
$\dfrac{p^{\frac{1}{2}}}{1+\sqrt{(\beta p)}}$	$\dfrac{1}{\sqrt{\beta}}\varepsilon^{t/\beta}\operatorname{erfc}\sqrt{(t/\beta)} = \dfrac{1}{\sqrt{(\pi t)}}\left[1-\left(\dfrac{\beta}{2t}\right) + 3\left(\dfrac{\beta}{2t}\right)^2 - 3{\cdot}5\left(\dfrac{\beta}{2t}\right)^3 + \dots\right]$
$p/\sqrt{(p^2+\alpha^2)}$	$J_0(\alpha t)$
$p/\sqrt{(p^2-\alpha^2)}$	$J_0(j\alpha t)$
$p/\sqrt{(\beta^2-p^2)}$	$(1/\pi)K_0(\beta t)$
$\dfrac{p}{[p+\sqrt{(p^2-1)}]^n\sqrt{(p^2-1)}}$	$I_n(t)$
$\dfrac{p}{[p+\sqrt{(p^2+1)}]^n\sqrt{(p^2+1)}}$	$J_n(t)$
$p/\sqrt{[(p+\alpha)(p+\beta)]}$	$\varepsilon^{-\frac{1}{2}(\alpha+\beta)t}I_0[\frac{1}{2}(\alpha-\beta)t]$
$\sqrt{[(p+\alpha)/p]}$	$\varepsilon^{-\frac{1}{2}\alpha t}[\alpha t I_1(\frac{1}{2}\alpha t) + (1+\alpha t)I_0(\frac{1}{2}\alpha t)]$
$\dfrac{\sqrt{(p+\alpha)}}{\sqrt{p}+\sqrt{(p+\alpha)}} - 1$	$-\frac{1}{2}\varepsilon^{-\frac{1}{2}\alpha t}[I_1(\frac{1}{2}\alpha t) + I_0(\frac{1}{2}\alpha t)]$
$p(p^2+\beta^2)^{-\alpha}$	$\dfrac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\dfrac{t}{2\beta}\right)^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\beta t)$
$[(p+\beta)^2+r^2]^{-\alpha}$	$\dfrac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\dfrac{t}{2r}\right)^{\alpha-\frac{1}{2}} \varepsilon^{-\beta t}J_{\alpha-\frac{1}{2}}(rt)$
$p\varepsilon^{-1/p}$	$-[J_1(2\sqrt{t})/\sqrt{t}]$
$\varepsilon^{-1/p}$	$J_0(2\sqrt{t})$
$(1/p)\varepsilon^{-1/p}$	$\sqrt{t}J_1(2\sqrt{t})$
$p\varepsilon^{-c\sqrt{(p^2+\alpha^2)}}$	$-\dfrac{\alpha c}{\sqrt{(t^2-c^2)}}J_1[\alpha\sqrt{(t^2-c^2)}]\quad (c<t)$
$p\varepsilon^{-\alpha\sqrt{(\beta^2-p^2)}}$	$\dfrac{\alpha\beta K_1[\beta\sqrt{\alpha^2+t^2)}]}{\pi\sqrt{(\alpha^2+t^2)}}$
$\dfrac{p}{\sqrt{(\alpha^2-p^2)}}\varepsilon^{-\alpha\sqrt{(\beta^2-p^2)}}$	$\dfrac{1}{\pi}K_0[\beta\sqrt{(\alpha^2+t^2)}]$

APPENDIX II

MATHEMATICAL FORMULAE

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + \ldots$$

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \ldots$$

$$\varepsilon^{\pm x} = 1 \pm x + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \ldots$$

$$\varepsilon^{jx} = \cos x + j \sin x$$

$$\varepsilon^{x} = \cosh x + \sinh x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh(x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$$

$$= \sqrt{(\sinh^2 x + \sin^2 y)} \, . \, \underline{/\phi_1}$$

$$\cosh(x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$$

$$= \sqrt{(\cosh^2 x - \sin^2 y)} \, . \, \underline{/\phi_2}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{x}{\sinh x} = 1 - \frac{2}{1 + (\pi/x)^2} + \frac{2}{1 + (2\pi/x)^2} - \ldots$$

$$x \coth x = 1 + \frac{2}{1 + (\pi/x)^2} + \frac{2}{1 + (2\pi/x)^2} + \ldots$$

$$\Gamma(1+n) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = n!$$

$$\Gamma(n) \, . \, \Gamma(1-n) = \pi/\sin n\pi.$$

INDEX

(Numbers refer to pages)